WHITEHEAD'S METHOD OF EXTENSIVE ABSTRACTION

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In ordinary parlance as well as in mathematics we have both intervals of space and (magnitudeless) points such that the extremities of some intervals, viz., the line-segments, are points. Now, there are three *prima facie* possibilities with regard to the question of primacy between intervals and points: (i) that the points are given and the intervals arise from them in some way (such as by summation or fluxion); (ii) that the intervals are given and the points arise from them in some way (such as by division or abstraction); and (iii) that both intervals and points are given independently of each other though certain relations bold between them as of necessity.

Euclid adopted, and most of us adopt, in actual practice, the third alternative. But, apart from defying Ockham's razor in a directional analysis of geometry, it involves an impossibility: if both intervals and points are defined completely independently of each other then no relations can hold between them as of necessity, and if certain relations do hold between them as of necessity then both of them are not definable completely independently of each other. Mathematicians were therefore obliged to choose either the first or the second alternative. They have adopted the first one, and Dedekind and Georg Cantor are taken to have actually derived lines, surfaces and solids from points and to have shown that, e.g., line-segments are nothing but nondenumerable sets of point-sets which satisfy the linearity conditions.

So far as I know, no one has adopted the second alternative with the possible exception of Alfred North Whitehead (1861-1947). Whitehead adopts a rather novel position —an amalgam of the firs' two alternatives: he takes the interval (the four-dimensional spatio-temporal event or region which is an ingredient of the physical world) as the most primitive geometrical element (in an extended sense, in a sense in which spatio-temporal intervals are geometrical elements) and derives what he calls a 'point' (which had better been called 'point-instant' instead), but goes on

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to derive what he calls 'lines', 'surfaces' and 'volumes' (which are one-three and four-dimensional spatio-temporal intervals). His position is perhaps best described as the first (i.e., mathematicians) alternative with this difference that as an empiricist he took the four-dimensional region as epistemologically more primitive than the point or point-instant. He may therefore he taken to have put himself to the task of abstracting his 'point' from the (undefined) notion of region (embedded in sense perception) with the help of the (undefined) relation of extensive connection (presumably also embedded in sense perception) and a number of self-evident propositions, and thus of endeavouring to establish geometry on a more secure epistemological foundation than had Euclid or even modern mathematicians.

In Section I we present a brief summary of Whitehead's method, and in Section II we defend the Method against Professor Grünbaum's objections, which serve as a useful introduction to Section III where we endeavour to show that the Method fails in deriving the point from the region, in deriving the line, surface and the volume from the point, and in defining the straight line, plane and the three-dimensional 'flat' locus. However, in Section IV we argue that, taken as an endeavour to adopt the second alternative, the Method was a pioneering, very useful, and highly commendable effort.

Whitehead has presented his method of extensive abstraction in four of his works:[1]

1. "La théorie relationiste de l'Espace", *Revue de métaphysique et de morale*, XXIII (1916), pp. 423-54.

2. *An Enquiry Concerning the Principles of Natural Knowledge*, Cambridge, 1919, Part 3.

3. *The Concept of Nature*, Cambridge, 1920, Chapter 4.

4. *Process and Reality*, New York, 1929, Part 4.

[1] In his preface (written in 1914) to *Our Knowledge of the External World* (first published in 1914, revised in 1926), Bertrand Russell says that he owed his definition of points and the treatment of instants to Whitehead and that what he had said on those topics in that book was in fact a rough preliminary account of the more precises results which Whitehead was giving in the fourth volume of their *Principia Mathematica*. The proposed fourth volume was never written. However, it is almost certain that Whitehead's exposition of his Method in that volume, had it been written, would have been about the same as he has given in the first two works listed in the text.

The method presented in all these works is essentially the same although there are some differences among them in matters of detail. It would in itself be of interest to study these differences and to trace the evolution of Whitehead's thought on and technique for extensive abstraction from his *Organisation of Thought* published in 1917 to the *Process and Reality* published in 1929. Some scholars have discussed some of these differences, but, I am not aware of any detailed study of those differences. However, in this article we propose to study the essential elements of Whitehead's method, since our main purpose here is to evaluate it, and shall therefore confine ourselves to only one of the four works, *Process and Reality*, which is Whitehead's magnum opus and contains his most mature attempt at extensive abstraction.

Whitehead takes *region* and *extensive connection* as indefinable terms and explains his usage concerning these two terms from which we learn that the former is at least a four-dimensional continuum and the latter means any kind of relation that any two regions can have to one another.[2] He first defines the concepts of inclusion or whole-part relationship, overlapping, dissection of a region (i.e., a set of mutually exclusive and collectively exhaustive parts), intersect of two regions (i.e., a region in which two regions overlap), unique and multiple intersection of two regions (if there are two or more non-contiguous intersects of two regions then the two regions have multiple intersection and if they have only one intersect then there is unique intersection), externally connected (i.e., contiguous), tangentially included (i.e., so contained that the part shares in the 'surface' of the whole) and non-tangentially included (i.e., so contained in the interior that the part in question is completely surrounded by another part of the given region), and then introduces the notion of an *abstractive set* as a set of regions any two of which are such that one of them includes the other non-tangentially and there is no region whatever which is included in every member of the set.[3] Thus, he presents the notion of convergence to a geometrical entity-point, line and, surface *without* postulating any of these entitles. That is, we begin with a region R of any size and then take as a member a region M which is non-tangentially included in the given region, i.e., a part of region R which is surrounded on all sides by another part of R having some thickness so that the upper surface of M

[2] A.N. Whitehead, *Process and Reality* (corrected edition, edited by D.R. Griffin and D.W. Sherburne), New York, 1978, Paperback ed., 1979, pp. 294, 301 and 304. (Hereinafter the 1979 paperback edition will be referred to as *PR*.)

[3] *PR*, pp. 295-98 (Definitions 2 to 10).

is not connected with any region not included in. By taking smaller and smaller such parts of R as the members of our set, we obtain a set of regions none of which is the smallest member and the regions converge to a surface, line or a point. (Even in the case of converging to a region, that is, to a three —or a four— dimensional continuum, when we begin with a hollow region, it is clear that such a region is not a member of the set but lies beyond 'all' the members of the set, just like the x + 1th *(omega-plus-one th)* member of an infinite convergent series, the members of the set approaching it more and more closely as we move down the converging end of R.)

Whitehead introduces the notion of one abstractive set *covering* another abstractive set (i.e., that of every member of one set including some member of the other) and that of 'equivalence of abstractive sets', or, in ordinary parlance, the notion of *sameness of convergence.*[4] A *geometrical element* is now defined as a complete group of equivalent abstractive sets, equivalent to one another and to no other abstractive set outside the group.[5] Then the notion of one geometrical element *being incident in* another geometrical element is introduced: when every member (abstractive set) of a geometrical element *a* covers every member of another geometrical element *b*, but *a* and *b* are not identical then *b* is said to be incident in *a* (i.e., to be contained in *a*).[6] And now we reach the 'point' as a geometrical element in which no other geometrical element is incident.[7] Whitehead points out this definition is to be compared with the Euclidean definition of a point as that which has no part.[8]

Now the notion of a geometrical element *being prime in reference to assigned conditions* is introduced by which Whitehead means that no other geometrical element satisfying those conditions is incident in the given geometrical element.[9] Whitehead points out that a point is an absolute prime in the sense that no other point or geometrical element can be incident in it.[10] He is now in a resition to define a *segment* as a geometrical

[4] *PR*, p. 298 (Defs. 11 and 12).

[5] *PR*, pp. 298-99 (Def. 13).

[6] *PR*, p. 299 (Def. 15). The use of the word 'identical' is a slip of the pen; it should have been 'equivalent'.

[7] *PR*, p. 299 (Def. 16).

[8] *Ibid.* But this remark should have been given as a separate paragraph by way of an observation on Def. 16.

[9] *PR*, p. 299 (Def. 16.1). I feel that it should have been numbered 17 instead of 16. 1, since it is quite independent of Def. 16.

[10] *PR*, p. 299.

element between points p and q in which p and q are incident and in which no geometrical element is incident in which also p and q are incident; p and q in such cases are to be called the *end-points* of the segment.[11]

Whitehead now introduces the notions of a point *being situated in* a region and in the surface of a region: a point is situated in any region which is a member of one of the abstractive sets composing that point, and a point is situated in the surface of a region x when all the regions in which that point is situated overlap with x but are not included in x.[12]

A complete locus of points can now be defined: A *complete locus of points* is a set of points that compose all the points situated in a region, or in the surface of a region, or all the points incident in a geometrical element.[13] The volume of a region is a complete locus consisting of all the points situated in that region; a surface of a region is a complete locus consisting of all the points situated in the surface of that region; and, a linear stretch between two end-points is a complete locus consisting of all the points incident in the segment between those two points.[14] Any complete locus of points consist of an infinite number of points.[15]

Whitehead makes an important remark about the Euclidean definition of a straight line. He says that the weakness of this definition is that nothing has been deduced from it whereas the uniqueness of a straight segment between two points (i.e., there being one and only one straight segment between any two points) should be deducible from it. Consequently, in modern times, as Whitehead points out, a straight line segment has been defined as the shortest distance between two points, and shortest distance has itself been practically defined as the line which is the route of certain physical occurrences. Whitehead tries to remedy this gap in the classical theory.[16]

Whitehead mentions a class of *oval* regions and says that it is to be defined. The only weapon that he finds for this definition is the notion of regions which overlap with a unique intersect. He says that evidently it is a property of a pair of ovals that they can only overlap with unique intersection, but, he says, it is equally evident that some non-oval regions

[11] *PR*, pp. 299-300 (Defs. 18 and 19).
[12] *PR*, p. 300 (Defs. 21 and 22).
[13] *PR*, p. 300 (Def. 23).
[14] *PR*, p. 300-301 (Def. 24).
[15] *PR*, p. 300 (Assumption 27).
[16] *PR*, p. 303.

also overlap with unique intersection. However, he says, the class of ovals has the property that any non-oval region overlaps some oval regions with multiple intersection. He admits that a single oval region cannot be defined but a class of oval regions can be defined inasmuch as a class can be defined whose members have to each other and to non-oval regions the properties ascribed by him to the class of oval regions. Such a class, he says, will be called ovate.[17]

Whitehead proposes a preliminary definition: An ovate abstractive set is an abstractive set whose members all belong to the complete ovate class under consideration.[18] He then defines an ovate class of regions as these which fulfil a certain group of non-abstractive and a certain group of abstractive conditions. The non-abstractive conditions are: (i) any two overlapping ovate regions have a unique intersect which also is an ovate region; (ii) a non-ovate region overlaps some ovate regions with multiple intersection; (iii) any ovate region overlaps some non-ovate regions with multiple intersection; (iv) the surfaces of any two externaly connected ovate regions teach either in a complete locus of points or in a single point; (v) the surface of a non-ovate regions touches the surface of some ovate region externally connected with it in a set of points which does not form a complete locus (i.e., the two regions touch in a set of points which does not comprise a line-segment, surface or volume); (vi) the surface of an ovate region touches the surface of some non-ovate region externally connected with it in a set of points which does not form a complete locus; (vii) any finite number of regions are included in some ovate region (i.e., there is a sufficiently large ovate region to *contain* any given finite number of regions); (viii) if A and B be any two ovate regions such that A includes B then there is an ovate region C such that A includes C and C includes B, and (ix) there are dissections of every ovate region which consist wholly of ovate regions, and, there are dissections which consist wholly or partly of non-ovate regions. The abstractive group of conditions are: (i) there are ovate abstractive sets among the members of any point; (ii) if any set of two, or of three, or of four, points be considered, there are ovate abstractive sets prime in reference to the condition of covering those points; and, there are sets of five points such that no ovate abstractive set is prime in reference to the condition of covering those points.[19] Whitehead points

[17] *Ibid.*
[18] *PR*, pp. 303-4 (Def. 0.1).
[19] *PR*, p. 304 (Def. 1).

out that by reason of the definitions of the abstractive group of conditions, the extensive continuum in question is four-dimensional.[20] An extensive continuum of any number of dimensions can be defined analogously.[21] Whitehead asks us to notice that the property of being dimensional is relative to a *particular* ovate class in the extensive continuum (emphasis ours): there may be ovate classes satisfying all the conditions except the dimensional conditions. He further informs that a continuum may have one number of dimensions relatively to one ovate class and another number of dimensions relatively to another ovate class. Whitehead opinies that the physical laws which presuppose continuity, possibly depend on the interwoven properties of two or more *distinct* ovate classes (emphasis ours).[22]

Whitehead assumes that there is at least one ovate class in the extensive continuum of the present epoch which has the two groups of characteristics enumerated above. He selects one such ovate class and says that all [further] definitions will be made relatively to the selected ovate class. He assures us that there being an alternative ovate class is immaterial to the argument; if there be such an other one, the derivative entities defined in reference to this alternative class are entirely different to those defined in reference to the selected class.[23] He now presents the theorem which is going to help prove the uniqueness of a straight segment; if two abstractive sets are prime in reference to the same two-fold condition of covering a given group of points and of being equivalent to some ovate abstractive set, then the two abstractive sets are equivalent.[24] He offers an elegant proof.[25] It follows as a corollary that all abstractive sets, prime with respect to the same two-fold condition of this type, belong to one geometrical element.[26]

We now come to the definition of a straight segment. If two abstractive sets are prime in reference to the same two-fold condition of cover-

[20] *PR*, p. 304.
[21] *PR*, pp. 304-5.
[22] *PR*, p. 305.
[23] *Ibid.*
[24] *Ibid.* (Assumption 2).
[25] *PR*, p. 305 (Proof of Assumption 2). In this proof, however, he says that regions MN *intersect* instead of saying that they *overlap* another slip of the pen. ('To overlap' has been defined but not 'to intersect'. *An* intersect has also been defined, but from its definition one cannot go on 'to intersect'.)
[26] *PR*, p. 305.

ing a given set of two points and of being equivalent to some ovate abstractive set, then the two sets are equivalent and belong to one geometrical element; this geometrical element is called a straight segment.[27] As can be readily seen, this definition itself shows the uniqueness of a straight segment. A similar definition is given of a flat geometrical element: instead of having two, we now have more than two, points.[28] Whitehead observes that straight segments are also included under the designation of flat geometrical elements.[29]

Realizing that it may so happen that the same geometrical element is definable by some sub-set as is defined by a given set of points, Whitehead offers a definition and a postulate to meet this difficulty: A set of points which defines a flat geometrical element is said to be in its lowest terms when it contains no sub-set defining the same flat geometrical element; and, no two sets of a finite number of points, both in their lowest terms, define the same geometrical element.[30]

Whitehead defines a straight line between two given points as the locus of points incident in a straight segment between those points.[31] (A straight segment between two given points was defined as a certain geometrical element. Now, a straight line between two points is being defined as a certain locus of points.) Similarly a flat locus is defined as the locus of points in flat geometrical element.[32] He relates a given flat locus with a section thereof through the assumption that if any sub-set of points lies in a flat locus, that sub-set too defines a flat locus contained within the given locus.[33] Now a complete straight line is defined as a locus of points such that (i) the straight line joining any two members of the locus lies wholly within the locus, (ii) every sub-set in the locus, which is in its lowest terms, consists of a pair of points, and (iii) no points can be added to the locus without loss of one, or both, of the characteristics (i) and (ii).[34]

Whitehead defines a triangle as the flat locus defined by three non-collinear points; these points are the angular points of the triangle.[35] A

[27] *PR*, p. 306. (Def. 3).
[28] *Ibid.*
[29] *Ibid.* This observation, I feel, should have presented as a separate paragraph.
[30] *PR*, p. 306 (Def. 4 and Assumption 3).
[31] *PR*, p. 306 (Def. 5).
[32] *PR*, p. 306 (Def. 6).
[33] *PR*, p. 306 (Assumption 4).
[34] *PR*, p. 306 (Def. 6.1).
[35] *PR*, p. 306 (Def. 7).

plane is defined as a locus of non-collinear points such that (i) the triangle defined by any three non-collinear members of the locus lies wholly within the locus, (ii) any finite number of points in the locus lies in some triangle wholly contained in the locus, and (iii) no set of points can be added to the locus without loss of one, or both, of the characteristics (i) and (ii).[36] Similarly, a tetrahedron is the flat locus defined by four non-coplanar points which are the corners of the tetrahedron.[37] We now come to the definition of a three dimensional flat space. It is a locus of non-coplanar points such that (i) the tetrahedron defined by any four non-coplanar points of the locus lies wholly within the locus, (ii) any finite number of points in the locus lies in some tetrahedron wholly contained in the locus, and (iii) no set of points can be added to the locus without the loss of one, or both, of the characteristics (i) and (ii).[38]

It is imperative at this stage to point out that Whitehead's terminology and Method both are at first sight confusing, but a little reflection suffices to dispel the confusion.

Whitehead takes the term 'region' more or less as an indefinable term and explicates his use of this term by saying that regions are the relata which are related to one another by the [primitive and undefined] relation of 'extensive connection', or, in other words, as the kind of things between whom this relation holds. He also says that the *volume* is the inside of a region. He further says that in the application of his theory of extension 'to the existing physical world of our epoch, volumes are four- dimensional, and surfaces are three dimensional'. From these statements it appears as if Whitehead takes the notion of a four-dimensional spatio-temporal region as primitive, and seeks to define or derive the notion of a magnitudeless spatio-temporal element which he calls a *point* because of its basic similarity to a point of space. Hence, it seems that his 'abstractive sets' consist of an infinitude of four-dimensional spatio-temporal regions none of which is included in every other member and one of any two members is non-tangentially included in the other. But, Whitehead also says, 'By reference to the particular case of three-dimensional space, we see that abstractive sets can have different types of convergence. For in this case, an abstractive set can converge either to a point, or to a line, or to an area'. This confuses the whole issue and one is at a loss to decide

[36] *PR*, p. 306 (Def. 8).
[37] *PR*, p. 306 (Def. 9).
[38] *PR*, p. 306 (Def. 10).

whether in a given context the term 'region' means a three-dimensional spatial region or a four dimensional spatio-temporal region, and whether the point derived from it is a point of space or an indivisible, magnitudeless spatio-temporal element.

When the matter is pondered, the realization comes that Whitehead takes his Method as a *general* method of abstraction in which one can begin with a 'region' of *any* number of dimensions, say n, and arrive at an indivisible magnitudeless element of the appropriate kind, and then from it build up the elements of one to n-1 dimensions the element of n-1 dimensions being the 'surface' of the given type of region.

It may also he pointed out that Professor Adolf Grünbaum's criticism, which we are going to discuss presently, is quite general and is applicable to both the abstraction of a point from the three-dimensional spatial region and the abstraction of a 'point' the n-dimensional 'region'.

However, in what follows, we are going to assume that a region is a three-dimensional spatial region, and, hence, that a surface is a two-dimensional spatial area, a line is a one-dimensional spatial extension, and, a point is a point of space. This will in no way vitiate the following discussion. For, whatever is seen to be true of the endeavour to abstract the point from the spatial region will hold *paripassu* of the attempt to abstract the 'point' from the four-dimensional space-time 'region'.

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Professor Adolf Grünbaum, one of the most notable philosophers of science of our times, subscribes to the first of the three views regarding the relation between the points and intervals of space, viz., that the points must be taken as given and the line-segments must be taken as nothing but non-denumerable sets of points which satisfy the memority conditions, and other geometrical entities as non-denumerable sets of points ordered in certain other ways. He is not happy with the idea of treating regious as given and seeking to abstract or derive the point from them, and, in a very influential article, he endeavours to show that Whitehead's attempt to derive the point from the region ended in a failure. On the contrary, Professor Grünbaum holds, Dedekind and Georg Cantor have succeeded in deriving the interval from the point and, having satisfactorily resolved the problems associated with the concept of the actually infinite have shown that the intervals are continua of non-denumerable sets of points. He also holds that Zeno's paradoxes stand resolved by these theories of infinity and continuity, provided that *continuousness* is not identified, as

Russell does, with mere compactness (there being an infinitude of points between any two points) which yields only a denumerable set of points having zero magnitude.

Professor Grünbaum holds Whitehead's method to have been a failure on two grounds. According to him, (i) the convergence of the abstractive sets or classes is fatally ambiguous,[39] and (ii) Whitehead's method is vitiated by one of Zeno's arguments.[40]

(1) Professor Grünbaum's first ground is valid in so far as Whitehead's earlier works, the *Enquiry* and the *Concept* are concerned. In those works, he had taken the expression 'A extends over B' to mean that B was a proper part of A. Now, if we take smaller and smaller (proper) parts of A as the members of an abstractive 'class', then, without appealing to the notions of a point, line, surface or volume, it cannot be determined as to what kind of an entity it is to which, does a given abstractive 'class' converge. For example, if we take an event E and wish to take out parts of E to converge to a line, but take out parts E_1, E_2, E_3... such that the 'surfaces' of E_1, E_2, E_3... have one and only one point in common, then the abstractive 'class' so obtained cannot converge to a line. Thus, to what an abstractive class converges was not determinable. Whitehead head had done nothing to forestall this ambiguity of convergence. And this ambiguity was fatal to his Method, since it entirely depended on the notion of sameness of convergence. In his *Process and Reality*, Whitehead removed this ambiguity by distinguishing between tangential and non-tangential inclusion and basing the notion of an abstractive set on that of non-tangential inclusion or non-tangential whole-part relationship. No two members of an abstractive set can now have a common outer surface or a common line-segment or point on their enter surfaces.

However, Professor Grünbaum holds that the Method even as presented in *PR* is beset by ambiguity of convergence. He asks us to take two distinct but neighbouring points such as $x = 10^{1000}$. It is clear that there is non-denumerable infinity of other points between the two chosen points. Now, Professor Grünbaum asks Whitehead to tell us (i) whether we know from sense perception that there exist two *different* abstractive classes defining those two points, and, if the answer be yes, to tell us (ii) as to precisely how their particular difference is certifiable by sense perception. Professor

[39] A. Grünbaum, "Whitehead's Method of Extensive Abstraction", *The British Journal for the Philosophy of Science,* IV, No. 15 (1953), pp. 215-26; see, pp. 219-26.

[40] *Ibid.*, pp. 216-19 and 222-26.

Grünbaum, it is submitted, does not see that a circularlity is involved in his rhetorical question, and that he is raising an irrelevant issue. He first asks us to assume that *there are* two points and then demands that their difference should be certifiable by sense experience. To be able to demand that the difference between two points should be demonstrable in sense experience, he would have to point out two perceptible things which can be represented by $x = 0$ and $x = 10^{1000}$. If he had succeeded in doing that, then Whitehead too would have succeeded in pointing to the perceptible difference between those two things. However, the point is that empiricism does not demand that everything we talk about should be perceptible. It would suffice if what we talk about can be brought into some intelligible relation with what is observable in sense perception. Hence, it is not required that we should be able to distinguish between two such points in sense perception so long as some rational principle can be laid down for the purpose of distinguishing the one from the other. If it were the case that we are unable to distinguish between two abstractive sets of regions, A and B, converging respectively to points $x = 0$ and $x = 10^{1000}$, then indeed Whitehead's method would have been fatally ambiguous and would have been a total failure on that account. But we see that B would have members (in fact, infinitely many members) which do not contain point $x = 0$ (that is, some members of B would not include any region which is a member of some set of regions that would ordinarily be said to converge to point $x = o$).

However, it seems to me that the convergence of the abstractive sets *is* ambiguous in one case, namely, in the case of a set that is supposed to converge to a *point* but which may only converge to a *surface*. That is to say, Whitehead does not provide a criterion to distinguish between those abstractive sets that would ordinarily be said to converge to a point and those that would ordinarily be said to converge to a surface.

Let there be an abstractive set that would ordinarily be said to converge to a sphere *s*. Let point p be the centre of sphere *s*. Now take a large sphere R concentric with and containing *s*. Spherical parts of R having p as their centre and larger than *s* would then constitute an abstractive set converging to *s*. Let us call this abstractive set A. It is clear that *s* is not a member of A: if we construct a set having as members R, R_1, R_2, R_3... such that p is the centre of each of these spheres and R_1 is contained in R, R_2 in R_1, R_3 in R_2, and so on, and such that each member is larger than *s*, then we have an infinite convergent series whose first member is R and *s* is in the nature of the (*omega-plus-one th*) term, i.e., *s* is the 'limit' of this series. We now take another abstractive set of regions B such that every member of B contains some member of A and similarly every member of A contains some member of B. It follows that B must also converge to *s*, for, otherwise, some member of A would fail to contain any member of B or some member of B would fail to contain any member of A. *Equivalence* of two abstractive sets (in Whitehead's sense) ensures *sameness of convergence*. Now, our objection is that having chosen the abstractive set A (and, consequently, set b as well as the 'complete' group of abstractive sets equivalent to A and to one another and equivalent to no other abstractive set outside the given group), if we were to assume that sphere *s* does not exist —that is, if we assume that R is a hollow sphere— then Whitehead's method partially fails, for now abstractive set A can only be said to converge to a surface, the surface of sphere *s*, but, Whitehead's method does not ensure that R must not be hollow, Whitehead simply assumes that a region (or solid) is not hollow. In other words, Whitehead should have made sure that something hollow cannot be taken to be a *region* but he failed to do so. However, this is not a crucial failure. The defect can be remedied by defining a *gap* and postulating that there are no gaps in any region. For example, Whitehead could have added two propositions at the end of Section II of Chapter II, Part IV (p. 297 of 1979 paperback edition) as follows:[41]

Definition 9 A. A region A is said to have no 'gap' in it when there are no two regions B and C such that A and B are a dissection of C, and C includes B non-tangentially.

Assumption 18 A. By 'region' we shall henceforth mean a region that has no gap in it. This assumption is merely a convenient arrangement of nomenclature.

It may moreover be pointed out that this was not a very important matter for Whitehead. For, his method was to jump from a (four-dimensional) region to a 'point' and build up a line, surface and solid from 'points'. A group of abstractive sets that is a 'point' can be unerringly

[41] I had originally worked probably with the New York, 1929 edition, and, it seems, it did not have the explanatory paragraph at the end of Section IV that we have in the 1979 corrected edition. In this paragraph Whitehead says that 'a certain boundedness is required for the notion of a region... The inside of a region... has a complete boundedness denied to the extensive potentiality external to it... Wherever there is ambiguity as to the contrast of boundedness between inside and outside, there is no proper region. This statement should have come in Section I of Chapter II. Even so, a further clarification should also have been made, that a region as thus conceived cannot be hollow from within, or have holes in it.

distinguished from any other group that is another 'point' or is a line, surface or solid, which is what alone matters.

(2) According to Professor Grünbaum, Whitehead's method is vitiated by Zeno's mathematical paradox of plurality. The argument, in its details, is somewhat as follows:

Part of the edifice of contemporary mathematics rests on the conception that a spatial interval is literally composed of unextended point-elements. But, obviously, no finite set of point-elements can add up to a positive interval, and as argued by Zeno (and demonstrated by Professor Grünbaum), not even a denumerably infinite set of point-elements can constitute a positive interval. A positive interval can only be constituted by a non-denumerably infinite set of point-elements. For Whitehead, a point is a (complete) group of abstractive sets of regions. Hence, metrical consistency demands that there should be a non- denumerable infinity of (groups of) abstractive sets of regions. Now, Whitehead's programme of epistemological reconstruction of geometry is that of beginning with something perceptible and by a process of abstraction arriving at things which are the termini of sense awareness. Hence, Whitehead's programme, in conjunction with the demand of metrical consistency, involves that there should be a non-denumerable infinity of abstractive sets and that these sets should be among the termini of sense awareness. Empiricists deny the existence of something actually infinite. Even if it is assumed that the existence of something actually, but only denumerably, infinite is certifiably by sense awareness, it is evident that the notion of actually infinite sets having a cardinality exceeding *aleph-null*, i.e., the notion of non-denumerably infinite sets, would inexorably defy encompassement by the sensory imagination. Hence, Whitehead's empirical programme is seen to be at variance with the demand of metrical consistency.

Professor Grünbaum expects this argument to demolish both Whitehead's method in particular, and the empiricist's aspiration to reduce non-empirical notions to empirical ones in general. Insofar as the latter expectation is concerned, it is quite unjustified. In the first place, an epistemological reconstruction of geometry along empiricist lines would begin by removing of geometry along empiricist lines would begin by removing from geometry the conception that supports part of the edifice of contemporary mathematics, viz., that an interval is constituted of magnitudeless elements. As such, no question of certifying the existence of a non-denumerable infinity of anything in sense experience or in anything else at all arises. In that case, the empiricists have of course to evolve points and instants, mass-points and particles, from phenomena that are perceptible, and would have to demonstrate that no illogicality was involved in such derivation. We believe that the empiricists' programme can be executed even though Whitehead may not have succeeded in evolving points from regions. (The notion of a point, we believe, is a rational notion. Hence, there must be a non-circular process through which human intellect arrived at the notion of a point. We have only to rediscover it consciously.) We are thus only left with the question of this argument's particular application to Whitehead.

Now, in relation to Whitehead, let it be noted that the argument involves both his derivation of the 'point' from the region and his derivation of the 'line', 'surface' and from 'points'. Insofar as his derivation of the point is concerned, this does not involve non-denumerable infinity, at least directly. However, if a spatial interval *is* constituted as modern mathematicians suppose it to be constituted. Then the 'complete' group of equivalent sets that is a geometrical element must have a non-denumerable infinity of members. But this should present no insurmountable difficulties since the abstractive sets would overlap with the members of the other sets. An abstractive set is not itself non-denumerably infinite, and, in fact, Whitehead asks us to think of them as a series of discrete members even though every one of them non-tangentially contains 'all' members coming after itself.

Insofar as Whitehead's derivation of the 'line' etc., from the 'points' is concerned, it is true that he does not explicity lay it down that only a non-denumerable infinity of points can constitute a line-segment, surface of a region, or a region. But he does not lay it down either that a positive interval is constituted only of a denumerable infinity of 'points'. Rather, since he uses the expression 'all points' he may be taken to have supposed a complete locus of points to be constituted of a non-denumerable infinity of points. Hence, if it be correct that Prof. Grünbaum's view succeeds in meeting Zeno's argument in question, then Whitehead too may be taken to have succeeded in meeting Zeno's argument. As for the claim that the existence of a non-denumerable infinity of abstractive sets. (Professor W. Mays says that it is by no means clear that Whitehead intended in epistemological reconstruction of geometry along empiricist lines, and, S. Nicod suggests, the Method may be considered after the fashion of an abstract mathematical model.[42] Had Whitehead had any such *reconstruc-*

[42] W. Mays, *Philosophy of Whitehead*, 1959, reprint, New York, 1962, pp. 113-14: "Whitehead does not always make it clear whether his method is to be taken as an algorithm or as an

tion at heart, he could hardly have tried to define lines, surfaces and volumes in terms of points. However, it is clear that he did not like to take the point as (intuitively) given and that he endeavoured to bring it into a rational relation with something sensible. Even so, this does not commit Whitehead to having a non-denumerable infinity of abstractive sets in perception.)

* * *

In any popular exposition of Whitehead's method, it is inevitable that the words "point", "line", and "surface" should occur before his definitions thereof occur, just as we had to do earlier. (Whitehead himself found it necessary, in an aside, to talk of convergence to a point before he had defined the point.[43]) This leads to the objection that a circularity is involved in the method. But, the fact is that the apparent circularity is involved only in the *exposition* of the method, not in the method itself. The definition of a point given by Whitehead does not presuppose the notion of a point: a point is a geometrical element in which no other geometrical element is incident, or, in other words, a complete group of equivalent abstractive sets in which no other complete group of equivalent abstractive sets in incident. And, as argued by Broad and Stebing, there is no circularity in popular expositions either, since 'convergence to a point' is itself understood in terms of regions and their relations.[44] (I am not happy with the actual defence through. But, we shall not argue this point since it relates only to popular expositions and not to Whitehead's method itself.)

Does our discussion in Section II lead to the conclusion that Whitehead's method is a sucess? We are afraid, it is not so. Rather, the fact is that his method does not succeed in abstracting the point from the region, i.e., in defining the term 'point' in terms of regions and extensive connections between them non-circularly.

Before we present the grounds for this statement, it is required to be very clear about one point, viz., that *an abstractive set does not converge to anything*, although it is quite natural for us, who assume that they know

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exact description of some actual process of convergence... Nicod... suggested that Whitehead's contribution could be taken as the c onstruction of a pure geometry rather than as an analysis of the real World."

[43] E.g., *PR*, p. 298.

[44] C.D. Broad, *Scientific Thought*, reprint, London, 1952, pp. 45-47; L.S. Stebbing, *A Modern Introduction to Logic*, reprint, London, 1958, pp. 446-52, esp. pp. 450-51.

what a point line or area is, to assume that an abstractive set must converge to a surface (an area), line or point.

Let us suppose that we select an abstractive set A by taking a large sphere s having point p at its centre and then an infinity of smaller and smaller spheres each having p at its centre. We see that the abstractive set A is converging to a point, and that the point to which the set of regions is converging is p. But, we do so only because we know (or suppose that we know) that there are points, that points are contained in regions, and that any given member region of A contains p as its centre. If we ask a boy, who has not yet been imparted the idea of a point, line or area, but who understands what a region is, to go through A starting from s, he will see that, there being no region at which he will be allowed to stop, he will have to ever remain engaged in the wearisome activity of mentally enlarging an unimaginably small region and taking a non-tangentionally included part of it for the same operation. But, even if he happens to select the spheres we had selected, he will not see spheres as converging to a point. As far as logic is concerned he will not be able to *reach* p, lost as he is in an infinity of operations. For the same reason, his attention will not come out of the regions and *arrive at* a point. An intuitive *jump* alone can enable him to arrive at the point. In mathematicians terminology: it is impossible to arrive at the point by taking a line and halving it into two, then halving one of the halves, and so on *ad infinitum*, because the point is in the nature of the w + 1 (omega + one) th term which remains at an infinite distance from any term no matter how far that term may be from the first term. Similarly, p is the w + 1 th term and cannot be arrived at by going through the concentric parts of s. In fine, the member regions of an abstractive set become smaller and smaller indefinitely, but do not converge *to* anything in the sense of *reaching* or *arriving at* the w + 1 th term, or in the sense of moving towards something ultimate, for, there is no logical means whereby it can be established that we are moving towards p, even though we may in fact be moving towards p. So far as logic is concerned, there may not be such a thing as a point, and, as far as the boy in question is concerned, there is no such thing as a point. The abstractive set of regions, if it were meant to convey the idea of a point, line or area, would utterly fail to convey such an idea.

It may be mentioned that at least in theory, Whitehead does not assume that the abstractive set should be supposed to converge to a point, etc. Hence, the fact that an abstractive set does not converge to a point, line or area, is no objection to Whitehead's method. The point, line or area is not defined by Whitehead as *that to which* a (complete) group of equivalest abstractive sets converge, but as that *group* of abstractive *sets* itself. In short, his method is based quite logically on *sameness of convergence*, or in his own terminology, on *equivalence* abstractive sets, and not circularly on that of convergence *to* a point, etc.

If one were to seek to take advantage of this fact and to hold that Whitehead's method is a failure because the equivalent abstractive sets *do not* converge to a point, line or surface, he would only be exhibiting his failure to understand the Method. In the face of this objection, Whitehead could have legitimately said that he did not at all postulate the entities ordinarily called point, lines, etc., and that he had no use for them and that it sufficed for his purpose that two abstractive sets had *sameness of convergence* even though neither converged *to* anything. If Professor Grünbaum were to insist that this would affect the continuity of the continuum, that if there were no surfaces, lines and points (as we understand these terms) then there would only be discrete regions, then Whitehead could say that he did not have to begin by assuming that spatio-temporal continua were continuous in the sense of there being boundaries between regions and that it would suffice for his purpose if regions were continuous in the sense that regions were contiguous and had no gaps in them. What is important for Whitehead is that the 'point' as defined by him does all the work that a point is required to do in geometry. However, it seems to me that (apart from the question whether Whitehead's point can do for our point) the fact that two abstractive sets have sameness of convergence but neither can be said to converge to anything (without already assuming that there are points, lines and surfaces and thus begging the question) presents at least an infelicity. (And this infelicity turns into a veritable perplixity when in popular expositions 'convergence to a point' etc., is glibly mentioned: convergence to a point *or* 'convergence' to a complete group of equivalent abstractive sets in which no such other group is incident, and if the latter then what does 'convergence to a certain group of abstractive sets' mean?)

The grounds on which we hold that the Method does not succeed in deriving the point from the region are as follows.

(1) Whitehead's notion of a region demonstrably presupposes the notion of a surface, a notion supposed to be defined in terms of regions, and, as such, the Method involves a circularity.

Whitehead begins with the notion of a region. He takes regions to be the relata of the (primitive) relation of extensive connection, or the sort of things among which the relation of extensive connection holds, and then seeks to define the notions of a point, line and surface. At the initial stage, the word "surface" is supposed to have no meaning, and we are supposed to have no notion of a surface, only that of a region. But, Whitehead's notion of a region presupposes the notion of a surface. It seems quite evident that what he really does is to enclose a portion of space by a surface which is then taken to be a region, and, what is more, he takes a surface to be complete, i.e., sufficient to enclose a region, and without any holes in it. This is evidenced by the following fact.

In the explanatory note appended to Chapter II of Part IV, Whitehead says that 'a certain determinate boundedness is required for the notice of a region' and further that '[t]he inside of a region, its volume, has a complete boundedness denied to the extensive potentiality external to it.'

If we conceive of a determinate portion of space, i.e., of a (spatial) region, then no doubt we can conceptually separate it from its neighbourhood. But, if the notion of a surface as something having no extension in one of the region's diremsions is not already given, it cannot be used to enclose a region. That is, the region, without the notion of a surface, can be quite determinate, but it is not obliged to comprehend all of the space that would ordinarily be said to be included in a region defined by a holeless and complete surface. We have seen that Whitehead assumes that there cannot be a hollow region, an assumption quite sufficient by itself to show that by the term 'region' Whitehead means a region enclosed by a holeless and complete surface.

(2) Whitehead's definiton of a point is infructuous. That is, his definition does not enable us to decide whether a given entity is or is not a point.

The abstraction of the point from the region depends on the notion of one abstractive set, A, *covering* another abstractive set, B, but Whitehead's definition of 'covering' does not enable us to establish whether A does or does not cover B for two reasons: (a) because of the infinitude of regions composing an abstractive set, and (b) because an abstractive set is not yet known to converge to anything.

(a) Any abstractive set consists of an infinity of regions. According to Whitehead's definition, set A would be said to cover set B when *every* member or A includes some member of B, i.e., when same member of B is a (proper) part of any given member of A. But, there is no way in which we can establish that every member of A, say, a, b, c, ..., includes some member of B: we cannot inspect each member of A individually to see whether or not it includes same member B because of their infinity, not does Whitehead prescribe any rule to establish generally that any member of the class of the orders of A must or does include some member of B. Nor is there any method to establish that every member of B is a (proper) part of some member of A for the same reasons, viz., because of the infinity of members of B, each of its members cannot be individually inspecied, and Whitehead does not offer any general rule to establish that any member of the class of regions constituting B must or is included in some member of A.

(b) If it were known that abstractive sets A and B both converge to, say, point p, then we could assert that any member of A must include some member of B and that any member of B must include some member of A. For, otherwise, A and B both could not converge to point p. If A converges to p and B converges to another point q, then there are regions containing p which do not contain q, and there are regions containing q which do not contain p, and, hence, A would have members which do not include any member of B, and B would have members which do not include any member of A. But, at the stage of deciding whether or not A covers B, we are supposed to have no ides of a point, and even if A is actually converging to p we are supposed to be ignorant of this fact. Thus, it is evident that Whitehead does not provide us with any means for establishing whether or not A covers B.

Now, as we have not been enabled to decide whether A does or does not cover B, it is evident that the definition of equivalence of abstractive sets is merely hypothetical, as opposed to categorical, and, as such, is of no use.

An abstractive set A is said to be equivalent to another abstractive set B when A covers B and B covers A. But, we have no means of establishing whether A does or does not cover b or whether B does or does not cover A. All that can therefore be said is that *if* A does cover B and B does cover A, *then* A and B are equivalent.

Since we have no means of establishing whether the abstractive sets A and b are equivalent, we have no means of arriving at a (complete) group of abstractive sets that are equivalent to one another and are not equivalent to any abstractive set not included in the given group. That is, if we have a group of abstractive sets given to us, then Whitehead's method does not enable as to decide whether or not it is a group of *equivalent* abstractive sets, or, in other words, we have not been provided with the means deciding whether or not a given group of abstractive sets is a point, line or surface.

A fortiori, since we have no means of arriving at a complete group of equivalent abstractive sets, we have no means of discovering whether one complete group of equivalent abstractive sets is or is not incident in another, whatever the term 'incident' day mean. Now, a complete group of abstractive sets M is said to be incident in another such group N when every member set of N covers every member set of M but no member set of M covers any member set of N. Since it cannot be decided whether any given member set of N covers any member set of M, no question of establishing whether M is or is not incident in N arises. Hence, it cannot be decided whether a given entitly is or is not a point. In other words, even if we chance to be presented with a complete group of abstractive sets (or a 'geometrical element' in Whitehead's terminology) in which no other geometrical element is incident —i.e., even if we chance to be presented with what is a point in Whitehead's terminology— we shall have no means of deciding whether the given entitly is or is not a goint.

We thus clearly see that Whitehead's definition of a point is infructious, and, as such, his method fails to define the point in terms of regions and extensive connections between them.

A person may, however, seek to defend Whitehead's method by saying that it does not matter that we cannot decide whether or not the abstractive set A covers the abstractive set b, all that we need are the *notions* of one set of regions covering another set, two abstractive sets being equivalent, and a geometrical element being incident in another, and once we have been imparted the notion of a point it should suffice for our purposes.

This, we feel, is not a good defence. The definition of an apple should enable us to decide whether or not a given entity is an apple, or, in other words, the definition should define appleness. If an attempted definition fails to capture appleness, and, as such, fails to enable us to decide whether a given entity is or is not an apple, then it is no definition. If the attempted definition of a point fails to enable us to decide whether a given entity is or is not a point, then it is evident that the definition has failed to capture or define point-ness.

(3) Whitehead's definition of a point as a certain *complete* group of equivalent abstractive sets is necessary but no group of equivalent abstractive sets can be complete.

The qualification of completeness is necessary because otherwise it would have been possible that a given group of equivalent abstractive sets is point p and another group of equivalent abstractive sets is point q but the members of p and q are equivalent and, hence, either a distinction would have to be drawn between the two groups, which seems impossible, or a rule would have to be laid down that p and q are the same, which in effect would amount to the completeness of the group.

The qualification of completeness is impossible of because no group of equivalent abstractive sets can be complete. It is evident that no finite collection of equivalent abstractive sets can be complete, since no matter how many such sets have been taken, there will still be some other set which is equivalent to each member of the collection but is not itself a member of this collection. The reason is that space is *ex hypothesi* infinitely divisible and hence given any two equivalent abstractive sets there is a third which is equivalent to both and in a sense lies between them. (Suppose we take set $S_1 = R_1, R_2, R_3, \ldots, R_n, \ldots$, and set $S_2 = E_1, E_2, E_3 \ldots, E_n \ldots$, such that R_1 constains E_1 and E_1 contains R_2, and so on. Then there is an abstractive set $S_3 = F_1, F_2, F_3 \ldots$ such that R_1 contains F_1, F_1 contains E_1 and E_1 contains R_2, and so on.) This means that given any abstractive set S, there are infinitely many abstractive sets that are equivalent to S. But there can be no infinite group or collection of anything, i.e., no determinate collection or group of anything can be infinite. (This point will be elaborated later in connection with the question whether 'an infinite set of points' has any meaning; please see sub-section ii.)

(4) Above all, Whitehead's 'point' does not answer to what we call a point.

We may not be able to state what we mean by the word "point" beyond what has been said by Euclid, but, I believe, we all mean the same thing (otherwise there would have been no geometry), and certainly what we mean by this world is not a complete group of equivalent abstractive sets of regions in which no other such group is incident (and whose member sets world ordinarily be said to converge to a point). C.D. Broad says that we must not be aghast at finding that the point had turned out to be different from what we had expected it to be.[45] Indeed, if we had supposed a ball to be made of iron and on analysis found out that it was made of silver, or we supposed the ball to be spherical and found out that it was made of silver, or we supposed the ball to be spherical and found out that it was oblong, then we ought not to be aghast at our finding. But, here

[45] *Scientific Thought*, p. 43.

we do not begin by assuming that the point is given and on analysis is discovered to be different from what we had expected it to be. Here, we believe we know what a point is and if we find that we are being presented with something different then we can at least say, "Well, your 'point' is different from ours". The crucial test here, as Broad rightly observes, is to see if Whitehead's 'point' can do for our point, and, we see that we cannot replace the definiendum (the ordinary word "point") in geometrical sentences by the definiens of Whitehead's definition of a point (a completed group of equivalant abstractive sets in which no other such group is incident).

This point may be seen in connection with the convergence to a point. We can easily understand the convergence of a pair of lines to a point, but we can make no sense of the convergence of two given complete loci of complete groups of equivalent abstractive sets to a certain understand what the word 'convergence' can mean in this context.

This point may be, further seen in connection with Whitehead's definition of being situated in a region. It is for us a truism that a point is situated in a region. But we do not comprehend what is meant when we are told that a certain group of equivalent abstractive sets of regions is said to be 'situated' in a region when that region is a member of one of the abstractive sets which compose that group of equivalent abstractive sets. Shorn of its technicalities, the definition tells us that a group of abstractive sets of regions is *situated* in any region which is a member of any of the abstractive sets of regions included in the group in question. We feel that 'to be situated in a region' as used by Whitehead does not mean what we mean when we say that a point is situated in a region. The gulf between the two usages appears to widen when a complete group of equivalent abstractive sets of regions is said by Whitehead to be situated in the surface of a region which is a member of one of the given abstractive sets of regions.

In short, a group of abstractive sets of regions is *not* a point (as ordinarly conceived) but merely a *route* or *pointer* to a point. It is unquestionably a better route or pointer than any that we have hitherto had, for example, better than the attempt to arrive at a point by dividing and subdividing a region. All the same, a group of abstractive sets is only a pointer or route to a point, not a point in itself. This, Whitehead had himself conceded in an earlier work, when he said:

There is no one event which the serios [of events forming an abstractive class] marks out, but the series itself is a route of approximation towards an ideal simplicity of content.[46] A route of approximation towards an ideal simplicity of 'content', it is submitted, is not itself an ideal simplicity of content.

We may put this argument as follows. If we knew what the word 'point' meant and were looking for point *p* then Whitehead's method would unerringly take us to point *p* and to no other point. That is, if we were in search of a route to *p* then nothing I know of could provide a better route to *p* than this method, for, *it is by taking p as the point of departure that the group of abstractive sets has infact (through not supposely in theory) been arrived at.*

However, if we are innocent of the notion of a point then despite guiding us towards point *p* by making sure that we do not by any chance wander on to any other point or to anything else of a different nature such as a line, Whitehead's method completely fails in yielding a point. What we have is a set of overlapping regions which become smaller and smaller indefinitely, and beckon a person wise to the situation towards *p* and leave an ignoramus like myself greatly baffled.

To sum it up, if we had to *represent* a point by something so that we could retain the distinction between points p_1 and p_2, then groups of equivalent abstractive sets could be used for this purpose: the distinction would be retained in as much as group g_2 cannot lead to p_2, g_1 being a route of approximation to p_1 and g_2 being a similar route to p_2. But if we desired to have something *equivalent* to what we call a point, or, what is the same, if we desired to learn what the word 'point' *means*, then the expression 'a complete group of equivalent abstractive sets of regions in which no other group of equivalent abstractive sets of regions is incident' is *not* equivalent to the word 'point', it does not tell us what a point really is. If so, Whitehead fails to *define* a point, and, *a fortiori*, fails to derive the point from the region.

(5) Whitehead jumps from the region to the point directly instead of deriving the surface from a region, a line from a surface, and a point from a line.

If Whitehead had succeeded in deriving the point from the region then this objection would have been pointless, although, even in that case, it would have pointed out an aesthetic infelicity.

[46] *An Enquiry Concerning the Principles of Natural Knowledge,* reprint, Cambridge, 1955, p. 104. In the *Concept of Nature* (reprint, Cambridge, 1971), Whitehead says, 'Thus an abstractive element is the group of *routes of approximation* to a definite in trinsic character of ideal simplicity to be found as a limit among natural facts.'

If fine, we see that in spite of taking advantage of the infelicity of jumping directly from the region to the point and of having sameness of convergence without there being a convergence *to*, Whitehead fails to find a non-circular method for defining the point.

In addition, it may be pointed out, Whitehead's method fails to derive the line from the point, if it is assumed that his 'point' is our point, and his 'line' is our line. (Otherwise, it would suffice to say that his 'point' and 'line' are not our point and line.)

Insofar as the derivation of the line, surface and volume is concerned, there is no difference between Whitehead and modern mathematicians -both derive the line, surface and volume from the point— and the arguments which can be urged against the one can be urged against the other.

(1) First of all, it seems strange that a magnitudinous whole should consist of magnitudeless parts. This difficulty is overcome by distinguishing between 'components' and 'constituents'.[47] Even so, it seems strange that a set of things each one of which is of zero magnitude should give rise to something that has positive magnitude.

Strange though it seems, this is what the mathematicians, Dedekind and Cantor in particular, are supposed to have succeded in doing. If S be a set of points such that for any value of x, if x is a point on line-segment *l* then x is a member of S and if there is no x such that x is a member of S but does not lie in l, then the members of S ordered in the manner they occur in l would be equivalent to l. Thus, all we need to do to dissolve the line-segment into a set of points is to find a set which has the property of set S. Now suppose that the line segment l is of the length of one centimetre. Let P_o be the first point or l, and p_1 be the last point of l. Now, any point p_n on l can be defined in terms of its distance from P_o: e.g., if p_n is at a distance of o.4. centimetres then we represent it by p.4. However, this is not sufficient to derive the line. We have to determine the relations that subsist between the points when they form a line. Dedekind and Cantor, therefore, endeavoured to determine what characteristics the supposed set of points S has. Now, the first characteristic of points is that no two points are consecutive. So, no two members of S may be consecutive if set out in the order of increasing (or decreasing) magnitude of their subscripts. Secondly, every point is an end-point of some sub-segment or l, and every

[47] C.D. Broad, *Scientific Thought*. p. 330.

sub-segment or l is such that an omega-sequence of points can be obtained having as its 'limit' the end-point of that sub-segment. So, every subset of S must contain a progression of members and the limits of such progressions must he members of the set S. Thirdly, if P_m and P_n be any two members of S and if m and n be rational numbers, then there must be a member of S, say Pr such that r is an irrational number greater than m and smaller than n, and conversely, if m and n are irrational numbers, then there must be a P_r such that r is a rational number greater than m and smaller than n. Given these conditions, to run through the members of S in the ascending order of magnitudes would be tantamount to running through l from P_o to p_1.[48]

Thus, the objection seems to have been overcome: we see how a line having a positive magnitude can be dissolved into points, or if you like, how can magnitudeless points give rise to a line.

However, it seems to me that the line is not done away with completely. Of course, the obvious objection that each point was defined in terms of its distance from a given point and that no definition of 'distance' in terms of points alone had been given, would be based on a mistake. In order to show that a line can be analysed in terms of points, the points, were initially defined in terms of distances, but once we see that an equivalence can be established between set S and line l, we can take the points independently of distances and in themselves: if the members of S have the three characteristics given above they give rise to a continuum of points. However, no rule appears to have been given to distinguish between the lengths of two continua of points. That is, since any continuum of points has a non-denumerable infinity of points, their magnitudes cannot be differentiated by the number of points. Indeed, in some cases, magnitudes of continua can be differentiated, e.g. where one is a part of the other, but, even in such cases, the ratios between the two can be worked out only by taking some continuum as the unit of comparison, which in effect means that some linesegment, in itself and quite independently of the points supposedly constituting it, would be adopted as the unit of measurement.

Mathematicians' inability to do away with the line completely is further exhibited by the phenomenon of motion. On the infinzer-atomicity

[48] R. Dedekind, *Essays on the Theory of Numbers* (tr. W.W. Beman), New York: Dover, n.d., esp. pp. 3-21; G. Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers* (tr. P.E.B. Jourdain), New York, 1915.

hypothesis adopted by modern mathematicians, as clearly avowed by Bertrand Russell, motion consists in being in different places at different times and in intermediate places at intermediate times, but there is no next place to be in at the next moment for the simple reason that there is no next place or next moment. Thus, in respect of a particle of matter, motion consists in its being in one point at one moment and in another point at a later moment. But the particle can never be in an adjacent point, for, no two points are adjacent. But, if so, how can the particle succeed in being in a different point at any later moment? The real answer is that the particle continuously moves along the line and thereby succeeds in being in a different point. The fact that there is an infinity of points between points p and q is of no help. It only helps in making the 'jumps' nearer; it does not enable the particle to be in an adjacent point. Thus, it is clearly seen that motion must consist in traversing the line and cannot consist only in being in different points at different moments. In other words, we do not succeed in resolving a line-segment into a set of points.

The fact that the line has not been completely done away with is also shown by the fact that the points are not supposed to constitute a discrete set, they are admitted by mathematicians to be a continuous set, or, in other words, no two points can be taken individually —they are the end-points of a line— segment.

(2) Moreover, there is a more fundamental objection to the mathematicians' assumption, namely, that there is no set of terms which could be the set S, or in other words, that 'set S' is not a 'set of terms' but only a *formula* for *generating* terms, and a formula which is in principle incapable of yielding any *given* set of terms. We have argued this point elsewhere;[49] here we present a summary of our argument.

That what we have called 'set S' cannot be a collection of terms is quite clear, since an 'infinite collection' is a contradiction in terms.

But 'set S' cannot be a *class* of terms either. It is true that the world 'class' is ordinarily used quite ambiguously so that we have both a defining property and the terms which have that property. And it is this practice which has given rise to the problem of universals. We are here using the world differently. We are so using the world that a given aggregation of terms each of the same sort or kind constitutes a *collection* and not a class, so that a class can stand in relation only to other classes and cannot

[49] F.A. Shamsi, "Infinzer-atomicity". *The Pakistan Philosophical Journal*, XIII, no. 3 (October 1975), pp. 47-84, and XIV, no. 2 (Jan.-June 1976), pp. 34-72.

be said to have an a number of members no matter what n may be and no matter how many entities be known to have the defining property of the given class. Moreover, the property that defines a class must be general and must not in any manner be restricted. That is, restriction on a class must come only from an additional qualification being imposed which must itself be general. Thus, there can be a class of animals and a class of points, but there cannot be a class of animals living in Pakistan or a class of animals existing in the 19th century; similarly there can be no class of points lying in this solid or that line-segment. For, 'living in Pakistan' or 'lying in that line-segment' are not general attributes or properties. If so, there can be no such class as the class of points between points p_m and p_n.

Let us suppose that 'set' means something different from a collection and a class. What will the expression 'all the members of S' now mean? If S were a *collection*, it would have meant x_1, x_2, x_3,..., x_n; but, S is not a collection. If S were a class, it would have meant the whole class to the exclusion of no sub-class; but S is not a *class*. What then can the expression in question signify? To me, it signifies nothing except the obstinate desire to do the impossible —to derive the line from the point.

(3) Finally, it appears to me that mathematicians took the wrong course in relating the line and the point: it is the point which is to be derived from the line and not the line from the point. Mathematicians thus not only reify the point, they completely fail to understand the nature of a point. A point is a potential division of a line just as a line is a potential division of a surface, and a surface that of a solid. To talk of all the points of l is thus to talk of all the divisions of l, and to equate a set of points with l is to equate a set of divisions or l with l and to hold that line-segment l is nothing but all the divisions or l. In a sense, the equation is true. If there is such a thing as 'all the divisions or l then no matter how disparate the category of 'divisions' and 'line-segments' may *prima facie* appear to be, nothing would be left in l if all its possible divisions were obtained. However, 'all the divisions or l', though it very much looks like 'all the boys in this room' has at best the same status as 'all men' and any attribute predicated of it must be analytic, i.e. the predicate must be a component of the complex of defining properties. But when we claim that 'all the divisions or l are given' then 'being given' does not at all seem to be a property of 'the class of divisions of l' (even assuming it to be a *class*).

Furthermore, Whitehead fails to define a straight segment. He defines a straight segment in terms of, *inter alia*, an 'ovate' abstractive set which he has not been able to define.

Whitehead begins by mentioning what he calls an 'oval' region and contrasts it with a non-oval region in a very vague and ambiguous fashion. He claims that it is *evident* that two (as yet undefined) oval regions *can* only overlap with unique intersection. I do not profess to understand what he means. In the literal sense of the word, a region would be called oval if it had the shape of an egg, and a region which did not have this shape, for example, a sphere, an obelisk or a pyramid, would be called a non-oval region. If so, why two oval and not two non-oval regions should overlap with unique intersection is by no means evident to me. Whitehead further says that any non-oval region overlaps some oval regions with multiple intersection, from which it appears as if some oval regions may not overlap any non-oval region with multiple intersection. Even so, we fail to have any definite idea of an oval region or of the distinction between an oval and a non-oval region.

Whitehead holds that a class of ovals can be defined although a single oval cannot be defined. It is submitted that this expression is logically inappropriate. An individual can be described, possibly, exhaustively described, but cannot be defined. A class of things can be defined but if a class is defined than every individual which belongs to that class can be distinguished from any other individual not belonging to that class. The cat called Pussey cannot be defined, it can only be described; the class of cats can be defined, which only means that cat-ness or the properties which a thing must posses in order to qualify to be called a cat can be exhaustively enumerated. Thus, if it were possible to define a class of ovals, then it would be possible to say what an oval was. But, Whitehead, in saying that a single oval cannot be defined, meant to say that it was not possible to state what characteristics a region must possess to be called an oval. If so, in a logically proper sense, it was not possible to define the class of ovals. Thus, we may take it that in claiming that the class of ovals was definable, what Whitehead really meant to say was that without defining the terms 'oval' and 'non-oval' a set of protocol propositions could be laid down stating relations between these terms which could lead us to divine in what senses the two terms might have been used.

Whitehead further confuses the issue by saying, "...we cannot define a single oval, but we can define a class of ovals. Such a class will be called 'ovate'." At first sight, this decision seems to be senseless: why not persevere with the term 'oval', why bring in yet another undefined term? But, on reflection, we see that Whitehead is not using the word "class" to mean things of the same kind in general, i.e., things having common charactistics whether or not there actually be a thing having the characteristics in question-in short, in a sense in which the notion of a null class is not a contradiction in terms. Hence, it would seem that by "ovate" he means that group of ovals which can be defined. This makes sense, but makes the notion of an oval even more confusing and out of our reach.

Coming to the ovate 'class', what Whitehead does is to tell us what relations two ovate regions must bear to one another, what relations an ovate region must bear to some non-ovate region, what relations a non-ovate region must bear to some ovate region, and that there are ovate abstractive sets. This is indeed no way of defining what an ovate region is. But, let us try to see what picture of an ovate region emerges from the protocol propositions.

First of all, an ovate region is not necessarily oval in shape. For, a sphere satisfies both the abstractive and non-abstractive conditions laid down by Whitehead. Going over the conditions of the two groups, we came to the conclusion that what Whitehead may have had in mind is what we may call a 'regular' region, i.e., a region bounded by a 'regular' surface and comprehending all that lies within that surface. In other words, a region having a surface free from all protuberances and depressions and whose interior is free from all gaps or hollowness. We arrive at this conclusion from the fact that two regular regions, neither of the two having any protuberance or depression, can overlap only in a single, continuous stretch, whereas a regular region with some non-regular region and a non-regular region with some regular region must overlap with multiple intersection. And the surfaces of any two regular regions must meet either in a point or in a continuous set of points, that is, in a line or a surface, whereas a regular surface and some irregular surface, and, similarly, an irregular surface and some regular surface, must meet in a non-continuous set of points, i.e., in a group of points which do not by themselves a line or a surface.

Although we cannot be definite that this is what Whitehead must have meant by an 'ovate' region, I feel that we cannot be far wrong in our belief, for, for purposes of extensive abstraction the notion of a regular region is indispensable. Hence, we may at least tentatively assume that by an ovate region Whitehead must have meant a regular region.

Now, if Whitehead did really mean by an ovate region what we have designated a regular region, then it is all too clear that, instead of endeavouring to determine the essential properties of a regular region and defining a regular region in terms of those properties, Whitehead only seized upon two characteristics of pairs of regular irregular regions/surfaces, namely, those of unique/multiple intersection and of intersecting in a group of points forming/not forming a line or surface, and tried to 'define' the regular region in terms of these two non-essential characteristics of pairs of regular/irregular regions/surfaces, i.e., characteristics which cannot be used to define the term 'a regular region', for, these properties characterize *relations* between *two* regions/surfaces, and, consequently, completely failed to define a regular, or in his own terminology, an ovate, region.

Thus, even though Whitehead's definition of a straight segment is such that the uniqueness of a straight segment is immediately deducible from the definition itself, which is clearly an improvement on the traditional treatment, this definition does not succeed in defining a straight segment since Whitehead had not succeeded in defining an ovate region even if he is regarded as having succeeded in telling us what he meant by an 'ovate' region. (It is to be noted that although our description of a regular region as 'a region whose surface is free from all protuberances and depressions and whose interior is free from all gaps or hollowness'' seems quite clear and intelligible, if the notion of a point has not already been defined, means nothing. To become meaningful, the words 'protuberance', 'depression', 'gap' or ''hollowness' will have to be defined without resorting to the notion of a point. When we try to do so, we find it very difficult even to distinguish between the surface and the interior of a region!)

Since, in our opinion, Whitehead has failed to derive the line-segment from the point and to define a straight segment, it follows that he has failed to derive the surface and volume from the point and has failed to define a plane, if 'surface', 'volume' and 'plane' mean what we mean by them; otherwise, his 'surface' 'volume' and 'plane' cannot do the worse our surface, volume and plane do in geometry.

Now that we come to the conclusion that Whitehead's method of extensive abstraction did not succeed in deriving the point, and in deriving the line and the surface, from the region (the latter two via the point), or in defining a straight line or a flat surface, must we regard this method as a historical curiosity, as yet another instance of an aberration of the kind human mind afford sample evidence of being prone to? I think that the answer is an emphatic "no".

Solutions of most philosophical problems have only been possible by the trial-and-error method after many false leads had been thoroughly worked through. When, finally, a definitive solution is arrived at, all the earlier attempts at polution are seen to be complements of the actual solution without which such a solution could hardly have been possible. Even though a failure in the ultimate analysis, the very fact that such an attempt was made is in itself of immense value. In attempting to derive the point from the region. Whitehead's method is on the right track: we are certainly not born with the notion of a point, and, hence, it is obvious that we acquire it by some such sub-conscious process as Whitehead's method. The final solution of this problem will be arrived at by the same rigorous logical method of beginning with a few underfined notions embedded in sense perception and a few universally acceptable axioms.

It is clear that the notions of tangential and non-tangential inclusion will prove helpful in any attempt at extensive abstraction. If the notions of point, line and surface are not given, then to be able to ensure that a given region is a plenum i.e., to ensure that a given outer surface encloses the entire region which would ordinarily be taken as enclosed within it the notion of non-tangential inclusion will be found to be of crucial importance.

The method of rigorous deduction, though not new with Whitehead, is of the greatest value and the only logical method for the derivation of the point from the interval. In relation to extensive abstraction, Whitehead's was the pioneering endeavour and will ever be a beacon to all those who might attempt extensive abstraction in the future.

Whitehead's procedure in defining a straight segment, that is, in offering a definition which shows the straight segment's uniqueness among the line-segments bounded by two given points was a wonderful attempt and one cannot but wish that it had succeeded. Whitehead had taken the property of being the shortest distance as the crucial defining property without falling a prey to the circularity involved in other attempts to define the notion of a straight line. It is clear that if the concept of straightness is ever to be caught hold of in a non-circular definition, that definition will have to be such that either the property of being the shortest distance between two points can be immediately deduced from the definition or the concept of being the shortest distance between two points can be defined with the help of the defined notion of a straight segment.

In short, we owe a debt of gratitude to Whitehead for his having attempted to derive the point by extensive abstraction from a datum which was a deliverance of the only primary source of human knowledge, sense perception.