AL-BÎRÛNÎ ON TRIGONOMETRY

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In his *Introduction to the History of Science* (vol. 1), Sarton attributed no trigonometric accomplishments to Al-Bîrûnî. In this paper, two trigonometric works by Al-Bîrûnî will be considered:

1. On chords in a circle, viz.: *Istikhrâj al-Awtâr fi'l-Dâ'irah.*

2. On the science of astronomy, viz.: *Maqâlîd ʿIlm al-Haiʾah.*

The first is on plane tirgonometry, and the second is on spherical trigonometry. In an attempt to discover any new development in these works, the history of trigonometry before Al-Bîrûnî is presented in brief.

It can be said that trigonometry, plane and spherical, was started by Hipparchus late in the second century B.C., as a subject in mathematical astronomy. Hipparchus made observations in Rhodes and Alexandria; in his records, a table of chords is made. This is the crude origin of table of sines; for chords he has $2\theta = 2r \cdot \sin\theta$. Hipparchus calculated chord-lengths by stereographic projection, showing knowledge of the theorem later stated and proved by Ptolemy who owes much to Hipparchus in his *Almagest,* despite the fact that they were separated by three centuries.

In between the two came Menelaos in the second half of the first century of our era. His *Spherics* is a treatise on spherical trigonometry as distinguished from stereography and astronomy. The work is in three parts; in the first, the spherical triangle is clearly defined, and the essential equations giving the relations between the sides and the angles are established.

In the *Almagest,* Books IX and XI give a splendid exposition of trigonometry, plane and spherical. The computing of chords for every 30 from 0 to 180° is made. Still chords, not sines, are used.

The first appearance of the sine functions took place in India, in the *Siddhântas*, works on theoretical astronomy, that started to appear in the 5th century A.D. These bear traces of early Greek influence, with original Hindu elementes, including the words "jya" and "jîva-barda"; these were

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arabicised into "jaib". Because the Arabic "jaib" means "pocket", the trigonometric "jaib" was translated into Latin as sinus.

In the *Paulisa-Siddhanta*, trigonometry is developed as a branch of geometry, using sines. A table of 24 sines is given in intervals of $3° 45'$ ($= 225'$), which is considered the greatest angle equal to its sine. The table is calculated by the rule $\sin(n+1)\alpha = 2\sin n\alpha - \sin(n-1)\alpha - \frac{\sin n\alpha}{\sin \alpha}$, where $\alpha = \sin\alpha = 225'$.

Now we turn to the Islamic era, starting with Al-Khwârizmî of the 9^{th} century, the first muslim to give a table of sines. Habash al-Hâsib of the 9^{th} century gave tables of sines and tangents. Al-Nairîzî made a systematic use of tangents. Al-Battânî wrote an elaborate astronomical treatise with an exposition of trigonometry having tables of sines, tangents and cotangents by intervals of 1° each, with the rule $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \alpha$.

In the 10^{th} century, Abû'l-Wafâ continued the elaboration of trigonometry. He was the first to show the generality of the sine theorem of the spherical triangles, giving a new method for calculating sines. $\sin 30'$ is given correct to 8 decimal places. The following are established:

a. Rules for $\sin(\alpha \pm \beta)$.

b. $2\sin^2 \frac{\alpha}{2} = 1 - \cos\alpha$

c. $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

d. Tables of secant and cosecant.

Abû Nasr, Al-Bîrûnî's teacher, worked also with trigonometric subjects. But Ibn Yûnus, of the 11^{th} century, is the greatest Muslim astronomer and trigonometrician. He used orthogonal projection in solving problems of spherical astronomy. He used a rule equivalent to our

$$\frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] = \cos\alpha\cdot\cos\beta.$$

and calculated sin 1° by the rule

$$\sin 1° = \frac{1}{3}\cdot\frac{8}{9}\sin\left(\frac{9}{8}\right)° + \frac{2}{3}\cdot\frac{16}{15}\sin\left(\frac{15}{16}\right)°$$

Kushyâr ibn Labbân must also be mentioned for continuing the efforts of Abû'l-Wafâ on developing the tangent function.

But all in all, Abû'l-Wafâ stands as the greatest trigonometrician in the Islamic era so far as we can now judge.

THE TRIGONOMETRY OF AL-BÎRÛNÎ

In 1927, C. Schoy wrote about the trigonometry of Al-Bîrûnî as found in his Canon, namely *Al-Qânûn al-Mas'ûdî*. In 1971, E.S. Kennedy wrote six pages in *Journal of Near Eastern Studies* about Al-Bîrûnî's *Maqâlîd 'Ilm al-Hai'ah,* which is on spherical trigonometry. His work on plane trigonometry, viz. the work on chords, is not unknown to the scholarly world. It covers parts of the famous Haidarabad collection called Rasâ'il al-Bîrûnî. This part, as it stands in the collection, contains with it several extraneous pages related to works by Ibrâhîm ibn Sînân; they interrupt the context and make it difficult to follow up Al-Bîrûnî's work. Yet that same part was studied in Russian, and even published in Egypt, apparently without noticing the extraneous stuff. However, early in the forties, I published a note in the Islamic Culture in which I pointed out the works of Ibn Sinân intermingled with those of Al-Bîrûnî. Several years later, I obtained a copy of the Haidarabad collection, which enabled me to put the works of each author together. I obtained the Leiden copy of Al-Bîrûnî's work on chords, but I found it to be a much abbreviated copy of the original.

With all this stuff at hand, a student of mine wrote an unpublished M.Sc. thesis on Al-Bîrûnî's *Istikhrâj al-Awtâr,* in Arabic. The following lines give a brief account of this work by Al-Bîrûnî:

I. Al-Bîrûnî's *Istikhrâj al-Awtâr.*

Early in the work, Al-Bîrûnî, points out that to use sines instead of chords, and to take the radius equal to unit length make the rules of trigonometry easier. But, apparently in agreement of with the common practice following Greek tradition, he puts these two ideas aside and uses chord-lengths with diameter (d) of any length. However, it helps to evaluate his results, if we bear in mind that:

i. $\text{chd}\theta = 2r \sin \theta/2$, and

ii. $\sin (90° - \theta) = \cos \theta$, (i.e. the sine of the complement) $= \sqrt{1\text{-}\sin^2\theta}$

He starts by giving proofs to four theorems which he takes as basis to his exposition. The theorems are as follows, in modern terms:

Given two chords in a circle, AB>BC, D being the midpoint of arc ABC, and DH⊥AB, then:

Theorem I: (fig 1)* AH = HB + BC, i.e., the perpendicular from the mid-point D to the broken line ABC, like that on the straight line, bisects the line. 25 proofs are given to this theorem, attributed to several people, Arabs, Persians and Greeks, including Archimedes and Serenus, an Egyptian Greek of the 4th century A.D. To this Serenus Al-Bîrûnî attributes a book called *Fî Uṣûl al-Handasa*, On the Elements of Geometry. We know, however, that Serenus worked on conic and cylindrical circles; no work on the elements of geometry is attributed to him anywhere else, as far as we know.

Theorem II. $\overline{AD}^2 = \overline{DB}^2 + AB{\cdot}BC$. (See, fig. 2).

This holds also if ABC is a straight line. 12 proofs are given to this theorem, four being by Al-Bîrûnî himself.

Theorem IV. $\Delta ADC - \Delta ABC = DH{\cdot}HB$. (See, fig. 4).

Three proofs are here given.

Theorem III. The data is here changed: AD = DC as before, but chord CB is out of the segment ADC. (See, fig. 3).

Here $\overline{DB}^2 = \overline{DC}^2 + AB{\cdot}BC$. Three proofs are given as well

In his introduction to this work, Al-Bîrûnî states that Al-Râzî had objected to the giving several proofs for each theorem. He replies angrily that his giving proofs attributed to mathematicians past and present is sort of a reminiscence that pleases him and them. As for us, we may respect Al-Râzî's point of view; but thanks are due to Al-Bîrûnî, because the mathematicians he mentioned, and/or their solutions, are otherwise unknown to us.

It is worthwhile pointing out that some of his own solutions are the shortest and easiest.

Five geometric constructions follow:

1. From two given points: A, B, to draw two lines: AC, BC, so that $\measuredangle ACB$ is equal to a given angle, and AC + BC is equal to a given length.

He gives a method which he describes as easier than those of Menelaus, Ibn Quarra, Abû'l-Jude, and not longer than that of Al-Sijzî.

2. The above data, but here AC – BC is equal to a given length. Two methods are given.

3. The above data still, but AC·BC is equal to a given area. Two methods are given.

4. Here AC:BC is equal to a given ratio.

* See the end of article for figures.

5. In a given circle, to construct a triangle so that its circumference is equal to a given lenght.

Proofs of the following are now given:

a. To find out the lengths of the perpendiculars from the vertices to the opposite side of a triangle of given sides.

He gives a proof by Archimedes an adds some constructions that make it shorter.

b. To find the area of a triangle, given its side-lengths. A proof by Archimedes is given.

c. To find the area of a cylic quadrilateral, given its side-lenghs. He gives an Indian method quoted by a certain 'Abdullâh al-Shannî.

Several astronomical problems and their solutions follow in application of the above theorems. They end with a problem said to be common in books of algebra: Two birds see a fish from the top of two trees of given heights; they dash at it with equal speed, but reach it at the same time; the trees are on opposite banks of a river of given breadth; required to find the position of the fish between the trees.

The Rules of Plane Trigonometry.

The above are introductory to plane trigonometry and used to derive its rules. Here I state again that chd $\theta = 2r \sin \theta/2$. For θ Al-Bîrûnî uses $1/\alpha$, which means $1/\alpha$ of a complete circuit. This 1/6 stands for 1/6 of 360°, i.e. 60°. The rules he gives are the following:

1. chd $1/6 = r$. This means to us that chd $60° = 2r \sin 30° = r$. Thus $\sin 30° = 1/2$.

This also yields $\cos 30° = \sin 60° = \sqrt{1-(1/2)^2} = \sqrt{3}/2$.

2. chd $1/10 = \sqrt{5r^2/4} - r/2$ From this we can derive that

$$\sin 18° = \frac{1}{4}(\sqrt{5}-1),$$ and thus $\cos 18° = \sin 72°$ can be derived.

3. chd $(180° - \theta) = \sqrt{\alpha^2 - \text{chd}^2\theta}$ which yields $\cos^2\theta = 1 - \sin^2\theta$.

4. A rule for chd 2θ, which yields $\sin 2\theta = 2 \sin\theta \cos\theta$.

5. A rule for chd $\theta/2$, which yields $\sin \theta/2 = \sqrt{1\text{-}\sqrt{\cos\theta/2}}$

Thus rules 4 and 5 make it possible, theoretically, to derive the sines and cosines of the successive doubles and successive halves of 30° and 18°.

6. chd $1/8 = \sqrt{2r^2 - 2r \text{ chd } 1/4,}$ which yields $\sin 22\frac{1°}{2} = \frac{1}{2}\sqrt{2\text{-}\sqrt{2}}.$ and $\sin 45° = 1/\sqrt{2}$.

7, 8. chd $(\alpha \pm \beta)$. These we may interpret as $\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$.

Thus with sin 18° and 15° given, sin 3° and cos 3° can be derived, and therefore the sines and cosines of their successive doubles and halves.

9. chd $\frac{1}{2}(\alpha + \beta)$ is also derived.

Thus a long table of sines and cosines can be formed to the degree of accuracy that the square root arithmetical operation can yield. But long as it may be, this table, built upon the above nine rules, does not yield chd 1°.

We must add here that most of the proofs of the above rule are his; the few that are not his are attributed to Ibn 'Irâq.

To find chd 1°, we have first to find out geometrically, he says, $\theta/3$, where chd θ is known. But nobody has been able to trisect the angle. Some have done so by approximation. He adds that Al-Kîndî, the philosopher, did that by instrumentation, whereas others of today use the properties of the hyperbola. All such methods do not give a steadfast arithmetical result. He proposes another approximation and adds that in his commentary on the *Zîj* of Ḥabash, he gave another approximation. To that, he says, he added the methods of all the ancients and the more recent mathematicians in his book called *Ḥaṣr al-Ṭuruq al-Sâ'irah fî Istikhrâj Awtâr al-Dâ'irah;* i.e., A Collection of Current Methods of Extracting the Chords of the Circle.

In conclusion, I may say that although he presents easy proofs, the rules of plane trigonometry he presents were discovered before him. Whatever credit that he rightly deserves, other than presenting easy proofs, is that he presents in his book plane trigonometry as a branch of mathematics seperated from astronomy.

The scribe ends by saying that Al-Bîrûnî completed this work on chords in the month of Rajab, 418 A.H. (8 or 9/1027).

II. Al-Bîrûnî's *Maqâlîd ʿIlm al-Hai'ah*

This is his work on spherical trigonometry. In Arabic, maqâlîd means keys, and 'ilm al-hai'ah is the science (mathematical) astronomy. Thus the title means the key-principles of mathematical astronomy. But the work is exclusively on spherical trigonometry, except for some application of the key-principles in astronomical problems.

I said above that Kennedy published a brief study of the work. I have recently learnt that a full study thereof has been published with a translation into French by Marie-Thérése Debarnot. The following lines give a summary of the spherical trigonometry in the work.

After a long introductory compliment to his then patron, Marzubân ibn Rustam ibn Sharwân, Al-Bîrûnî starts by saying that when great circles on the surface of a sphere intersect, they form angles. To find the unknowns in such circles, or their ratios to each other, we have to resort to the ratios between their sines, which requires resorting to the figure called al-qaṭṭâʿ, which is formed by arcs of great intersecting circles. Ptolemy mentioned this figure in Book I of the *Almagest*, and Menelaus mentioned it in his *Spherics*. But Al-Nairîzî and Al-Khâzin developed it in their commentaries on the *Almagest*. Al-Khâzin reproduced it in brief in his *Zîj al-Ṣafâ'iḥ*, and so did Ibn ʿIrâq in his *Tahdhîb al-Taʿâlîm*; Ibn Quarra put down a full book on this figure. Many of the later scientists, like Al-Baġhdâdî, Ibn ʿIṣmat, Al-Sijzî and others worked on his figure diligently.

But in the present era, in which we find many contradictory things, intensive scientific knowledge and ability to solve problems that the earlier ones failed to solve, is shadowed by jealousy and lack of objectivity. He goes on to present the characteristics of the figure, attributing every finding to the person who deserves it:

He says that Al-Sijzî had procured several methods, purtaining to geometricians and astronomers, for finding out the direction and azimuth of al-qibla by the aid of instruments, all giving different results and supported by no proofs. Al-Bîrûnî told him that his teacher, Ibn 'Irâq, might be able to derive a correct method supported by proof. By the request of Al-Sijzî, Al-Bîrûnî requested Ibn 'Irâq to solve the problem. The result was the book of azimuths, *Al-Sumût*, by Ibn 'Irâq in which the properties of al-qaṭṭâ' are presented by the way. He who knows the complete quadrilateral in plane geometry may recognize this figure on the spherical surface; both are quadrilaterals with every two opposite sides produced to meet. The complete spherical quadrilateral is al-shakl al-qaṭṭa'.

Al-Bîrûnî continues saying that he had received a request from Abû'l-Wafâ al-Buzjânî to send him the book, which he received in Baghdad. He esteemed the book, but said that the author used old methods in presenting the properties of the figure, while he, i.e. Abû'l-Wâfâ, had easy methods for finding out the azimuth.

Both Al-Buzjânî and Ibn 'Irâq developed the point separately and informed Al-Bîrûnî about their findings. That knowledge he used to derive the methods he presents in his *Maqâlîd*. He goes on saying that, in Al-Ray, he met al-Khujandî who showed him a book of his in which al-qaṭṭa' was presented with new proofs. He also discovered that Kushyâr wrote a book in which al-qaṭṭâ' is almost all replaced by easier methods.

It seem that Abû'l-Wafâ claimed for himself parts of *Al-Sumût* of Ibn 'Irâq, while Al-Khujandî claimed for himself parts of the work of Abû'l-Wafâ, Kushyâr confessed that all he did was bringing together brief accomplishments made by others. Only Ibn 'Irâq, Al-Bîrûnî says, in unselfish and modest enough to overlook others claiming achievements he himself did.

Al-Bîrûnî now presents the theorems and proofs of the above-mentioned colleagues. These are as follows:

1. From a given point on one of two intersecting planes, two lines are drawn: One perpendicular on the other plane, and one perpendicular on the common line between the two planes; required to prove that the line joining the feet the two perpendiculars is also perpendicular on the common line.

The theorem and proof are attributed to Ibn 'Irâq; but Al-Bîrûnî adds another proof of his own.

2. A, B, C are two given points on a circle of diameter DB. If this diameter and line AC are produced to meet at H, then AH:HC = sin arc BA: sin arc BC.[1]

The theorem is attributed to Ptolemy, who must have used chords and not sines; A proof by Abû'l-Wafâ is presented.

3. The ratio between the sines of the sides of a spherical triangle is equal to the ratio between the sines of the opposite angles.

The theorem is attributed to Ibn 'Irâq; to his proof, Al-Bîrûnî adds another proof of his own. A second proof by Ibn 'Irâq is also presented.

4. Two theorems, with proofs, are given by Abû'l-Wafâ. This is the first: Two arcs of great circles on a sphere intersect making an acute angle; points are laid at random on the first arc: required to prove that the sines of the arcs between each of these points and the vertex of the acute angle are in the same ratio as that between their inclinations (the inclination of arc AB on arc AC is the arc from B perpendicular to AC).[1] Two proofs are given by Abû'l-Wafâ.

[1] These figures are missing in the English text as given to us by the late Prof. Saidan. Figures 1, 2, 3 and 4 were kindly supplied by his daughter, Amal Saidan.

Al-Bîrûnî adds a proof to this theorem of his own. This is the theorem called al-shakl al-mughnî, i.e., the figure that dispenses with al-qaṭṭâʿ.

5. The second theorem by Abû'l-Wafâ goes as follows:

In the mughnî figure, the arcs that the inclinations cut from the circle have their sines in the same ratio as the tangents of these inclinations.

Before that Al-Bîrûnî introduces what is meant by the tangent and the verse-tangent, and adds that to Abû'l-Wafâ, and all over in the book, wherever tangent comes, the verse-tangent is meant.

Here Al-Bûrûnî introduces the so-called ẓill al-miqyâs, i.e. tangent, tangent of the gnomon, and shows how to find it out. When the shadow of the gnomon falls on the ground, the tangent, i.e. the length of the shadow, is the plane tangent. If the shadow is vertical. its length is the verse-tangent (in fact, this verse-tangent is the tangent as we understand it; the plane tangent is what we now call the cotengent: It is the shadow of a horizontal line upon a vertical plane, whereas the plane tangent is the shadow of a vertical line upon the horizon. It should be pointed out that, in Arabic, both tangent and shadow are called ẓill).

6. This is the so-called al-shakl al-ẓillî, i.e. the tangential figure; it is attributed to Abû'l-Wafâ; it reads as follows:

If two great circles on a sphere intersect making an acute angle and points are put down on the arc of one of them, and arcs of great circles joining the two poles pass by these points, then the ratio between the sines of these arcs is equal to the ratio between the tangents of their altitudes. What Al-Bîrûnî calls altitude here is what Abû'l-Wafâ calls second inclination.

To the same theory Ibn ʿIrâq had given two proofs in his *Al-Sumût*. Another is given by al-Khujandî and a abridged one by Kushyâr. Al-Nairîzî, Al-Khâzin and Al-Bîrûnî himself, each gave a proof.

Spherical Triangles

Al-Bîrûnî starts now his presentation of spherical triangles by classifying them into the following ten kinds according to the angles between the arcs:

1) 3 acute angles,	2) 3 right angles	3) 3 obtuse angles,
4) 2 acute, 1 right,	5) 2 acute, 1 obtuse,	6) 2 right, 1 acute,
7) 2 right, 1 obtuse	8) 2 obtuse, 1 acute,	9) obtuse, 1 right,

10) 1 acute, 1 right, 1 obtuse.

Mention of the vertically opposite triangle on the sphere is made, showing that kind 1 is vertically opposite to kind 8, and so are kinds 3 and 5, and kinds 4 and 9. The remaining four are each reflexed into itself, i.e., the triangle vertically opposite to kind 2 is of kind 2, and so is each of kinds 7 and 8 and 10.

Thus when any triangle is solved, i.e., all its 6 elements are known, so is the vertically opposite one. Hence we need in fact to consider the solutions of kinds 1, 3, 4, 2, 7 and 8. Al-Bîrûnî starts with kind 4, namely a triangle having one right and two acute angles, like ΔABC, C being right.

Given three of its six elements known, a table is made to show in which case, the unknown three can be calculated. The solution depens upon the qaṭṭâ'. AC, AB are extended to D, E making AD, AE one quarter of a great circle each. With A as center, and straight line AD as radius, on arc DEZ is drawn, meeting arc CB extended at Z. Thus a qaṭṭâ' ADZB is formed. Similarly, qaṭṭâ' BHKA is formed. Thus DE = ∢A, EZ = 90°-∢A, TH∢B, TK = 90°-∢B; AE = ∢C = 90° (see, fig. 5).

With ∢C = 90°, if two sides, one side and one angle, or two angles are given, the triangle is solved, and all the qaṭṭâ' ADZB is known. The major formula is that the ratio of the sines of the sides ise equal to that of the sines of the angles, or the sines of their inclinations. The other different kinds of triangles, namely, 1, 3 and 10 are similarly considered.

The remaining pages of Al-Bîrûnî's *Maqâlîd* discuss astronomical problems using Abû'l-Wafâ's Mughnî figure. Examples of these problems are the following: To find out the inclination from the equater of the ecliptis, to find out the altitude of any degree, to find out ascensions, amplitudes, equation of the day, etc. These remaining pages form in fact more than half the full work, thus justifying its name which means that it is a book of astronomy.

One point remains to be mentioned in passing. The work called *Al-Ahilla*, by Ibn al-Haitham is in fact a study of al-qaṭṭâ' not mentioned by, and apparently unknown to Al-Bîrûnî. I believe that any comprehensive study of the Arabic achievements in spherical trigonometry should take Ibn al-Haitham's works amongst others mentioned by Al-Bîrûnî.

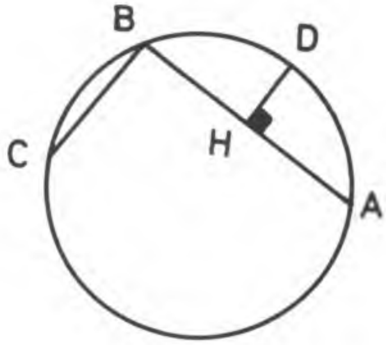

Fig. 1

Fig. 2

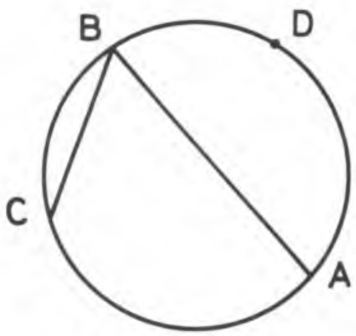

Fig. 3

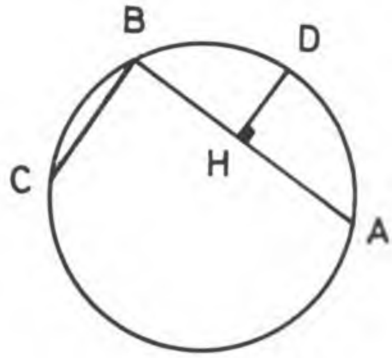

Fig. 4

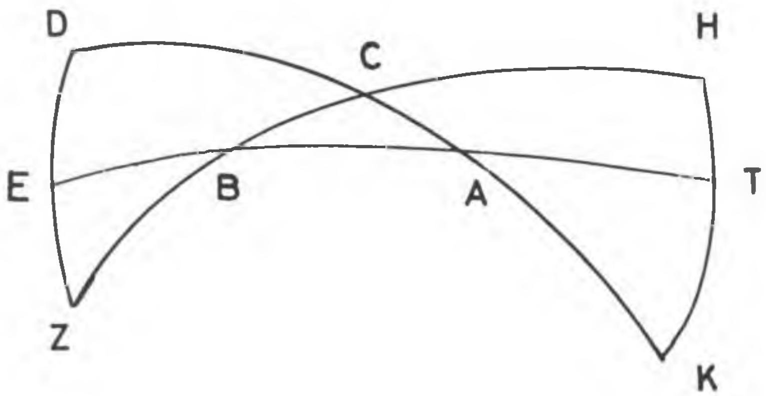

Fig. 5