THE FIRST PERFECT NUMBERS AND THREE TYPES OF AMICABLE NUMBERS IN A MANUSCRIPT ON ELEMANTARY NUMBER THEORY BY IBN FALLÛS

SONJA BRENTJES*

Several recent researches have revealed the intensive attention that was paid by the scientists of the Islamic world since the 9th century to elemantary number theory.[1] This interest started with the translation of the "Elements" of Euclid at the close of the 8th century and during the 9th century. In the beginning of the 9th century Ḥabîb ibn Bahrîz translated a Syrian paraphrase of the "Introductio Arithmeticae" by Nicomachus of Gerasa, which was commented upon by al-Kindî.[2] Some decades later Thâbit Ibn Qurra translated the complete text of the "Introductio Arithmeticae" directly from Greek into Arabic. Already in the 9th century the mathematicians added new results to the ancient knowledge. The most important 9th century treatise on elementary number theory was composed by Thâbit b. Qurra. It treats the construction of amicable, perfect, abundant and deficient numbers in a way, which surpasses the antique heritage. Its terminology resorts several times to the "Introductio Arithmeticae", while its methodology of proofs is based on the arithmetical books

* Dr., Karl Sudhoff Institut für Geschichte der Medizin und Naturwissenschaften, Karl Marx Universität, Leipzig.

[1] Matvievskaja, G.P., *Učenie o čisle na srednevekovom Bližnem i Srednem Vostoke,* Taškent, 1967; Matvievskaja, G.P., "Materialy k istorii učenija o čisle nasrednevekovom Bližnem i Srednem Vostoke," *Iz istorii točnych nauk na srednevekovom Bližnem i Srednem Vostoke,* Taškent 1972, pp. 76-169; Muzafarova, Ch. R., "Arifmetika Nikomacha v izloženii Kutbaddina Širazi," *Mat. i metodika ee prepod.*, cilt 1, Dušanbe, 1974, pp. 124-131; Muzafarova, Ch. R., Arifmeticeskie i teoretiko-čislovye aspekty knigi VII "Načal" Evklida v izloženii Kutbaddina Širazi," *Issledovanija po matematike,* Dušanbe 1977, pp. 79-84; Soussi, M., "Un texte d'Ibn al-Bannî sur les nombres parfaits, abondants, deficients et amiables," International Congress of Mathematical Sciences, July 14, 1975-July, 20, 1975, Hamdard National Foundation, Pakistan 1975; Rashed, R., "Nombres amiables, parties aliquotes et nombres figures aux XIII ème et XIV ème siecles," *Archive for History of Exact Sciences,* vol. 28, 1983 pp.107-147.

[2] UB Halle/S., Yb 5. 4°, ff 1^b — 54^a; See also: Steinschneider, M., *Die Hebraeischen Uebersetzungen des Mittelalters und die Juden als Dolmetscher,* Berlin 1893, vol. II, 320, p. 517.

of the "Elements".[3] Several scientists transmitted its content at least in part until the 17th century. As cursory examinations of Central Asian manuscripts show, this interest continued until the 19th century.[4] Among those who wrote on elemantary number theory are well-known Arabic-Islamic scholars such as al-Bîrûnî,[5] Ibn Sînâ,[6] Abû'l-Wafâ,[7] al-Karajî,[8] Quṭb ad-Dîn ash-Shîrâzî,[9] al-Kâshî,[10] and the author or the authors of the Rasâ il Ikhwân aṣ-Ṣafâ[11] as well as a multitude of other writers such as Abû Manṣûr b. Ṭâhir al-Baghdâdî,[12] Kamâl ad-Dîn al-Fâris,[13] Abû Saqr al-Qabîṣî,[14] and others.[15] Some of these writers like Abû Manṣûr al-Baghdâdî and Kamâl ad-Dîn al-Fârisî should be evaluated anew by historians of mathematics from the standpoint of their number theoretical works.[16]

[3] Saidan, A.S., *Amicable numbers by Thâbit ibn Qurra*, Amman (publication sponsored by the Jordan University), 1977.

[4] For example in: Bayyân aʿdâd. Rukopis'nyj fond' AN Tadžikskoj SSR, Dušanbe, Ms 1213, f 335*ff; Majmûʿe-ye risâlât. at the same place, Ms 3960; Majmûʿa rasâ'il arabîya. At the same place, Ms 4410.

[5] al-Bîrûnî, *Kitâb at-tafhîm li awâ'il sinâ'at at-tanjîm*, ed. J. Humâ'î, Tehran 1319 H, pp. 33-55; See also: Abu Raichan Beruni (913-1048), "Kniga vrazumlenija načatkam nauki o zvesdach," *Izbrannye proizvedenija*, vol. VI, Taškent 1975, pp. 38-50.

[6] Ibn Sînâ, *ash-Shifâ', al-fann ath thânî fi'r-riyâḍîyât, al-ḥisâb*, ed. ʿA.L. Mazhar, al-Qâhira 1975.

[7] Abuʿl-Wafâ' al-Buzjânî, *Risâla fi'l-aritmât*îqî, Rukopis'nyj fond' instituta vostokovedenija AN Uzbekskoj SSR, Taškent Ms 4750, ff 255^b-257^b; See also: "Traktat Abu-l-Wafy ob osnovnych opredelenijach teoretičeskoj arifmetiki." In: Matvievskaja, G.P., Ch. *Tllašev, Matematičeskie rukopisi učenych Srednej Azii*, X, XVIII vv. Taškent 1981, pp. 63-76.

[8] *L'Algebre d'al-Badiʿ d'Al-Karagî*, Edition, Introduction et Notes par A. Anbouba, Publications de l'Universite Libanaise, Section des Études Mathématiques, Beyrouth 1964.

[9] Quṭb ad-Dîn ash-Shîrâzî, *Durrat at-tâj lig hurrat ad-Dibâj*, ed. Mashkût, S. M., Tehran 1317-1320 *H.*

[10] Al-Kâshî, *Miftâh al-ḥisâb*, ed. an-Nabulsî, N., Dimashg 1977.

[11] *Rasâ'il Ikhwân as-Safâ'*, 1306 H.

[12] Abû Mansûr, Abd'l-Qahir ibn Tahir Al-Baghdâdî, *Al-Takmila fi'l-Hisâb* (The Completion of Arithmetic) *With a tract on Mensuration*, Edited and annotated with comparative by A. S. Saidan, Publications of Institute of Arab Manuscripts, Kuwait 1985.

[13] Kamâl ad-Dîn al-Fârisî, Tadhkirat al-aḥbâb fî bayyân at-taḥâbb, In: Rashed, R., "Matériaux Pour l'Histoire de Nombres Amiables et de l'Analyse Combinatoire", *Journal for the History of Arabic Science*, vol. 6, 1982, 209-278, pp. 266-229 (in Arabic)

[14] Risâlat al-Qabîṣî fî jamʿ anwâʿ min al-aʿdâd. In: Anbouba, A., "Un mémoire d'al-Qabîṣî (4e siècle H.) sur certaines sommations numériques," *Journal for the History of Arabic Science*, vol. 6, 1982, 181-208, pp. 201-187 (in Arabic)

[15] Compare Rashed, R., "Nombres amiables, parties aliquotes et nombres figurés aux XIIIeme et XIVème siècles", *op. cit.*, p. 109ff.

[16] Compare *ibid.*, pp. 111, 115, 122-147.

With the here summarized manuscript Ibn Fallûs becomes a member of this class.

Shams ad-Dîn Abû't-Țâhir Ismâîl ibn Ibrâhîm ibn Ghâzî ibn ʿAlî ibn Muḥammad al-Ḥanafî al-Mârdînî, called Ibn Fallûs, lived between 1194 and 1252. His epitome on elemantary number theory[17] was written during a pilgrimage to Mecca.[18] The author says that the "Introductio Arithmeticae" of Nicomachus was its basic source.[19] He follows Nicomachus in the classification of numbers, the majority of the properties of these numbers, and the philosophical back-ground; but he adds some new classes of numbers. He declines to discuss the theory of numerical ratios and proportions of Nicomachus, but he promises to deal with this subject in a separate work.[20] Whether he actually wrote such a treatise is uncertain.[21] There are no traces of number mysticism in the sense of the "Theologoumenates arithmetikes" in this compendium by Ibn Fallûs.[22] His epitome is a purely mathematical text, interspersed by some philosophical remarks imbedded in an Islamic context.[23] The treatise is devoted to the description of mathematical properties and principles of construction for the 25 kinds of numbers, that Ibn Fallûs defines. The tendency for completeness, however seems to encourage the author to leave the domain of mathematics by introducing some oddities. The "inimical numbers", about which

[17] Shams ad-Dîn Abu-ț-Țâhir Ismâ îl ibn Ibrâhîm ibn Ghâzî ibn ʿAlî al-Ḥanafî al-Mâridînî, *Kitâb iʿdâd al-isrâr fî asrâr al-aʿdâd, Staatsbibliothek, Preußischer Kulturbesitz*, Berlin, Ms 5970, Lbg. 199, ff 15ª-31ª; Dâr al-Kutub, Cairo, Ms B 23317, 3, ff 62ª-72ª; also: Aya Sofya, Istanbul Ms 2761, 7

[18] *ibid.*, Ms 5970, Lbg. 199, f 15ᵇ, 3.

[19] *ibid.*, f 15ᵇ, 11f.

[20] *ibid.*, f 15ᵇ, 12f.

[21] Aside from the number theoretical treatise there are four further works by Ismâʿîl ibn Ibrâhîm ibn Fallûs:

— Irshâd al-ḥussâb fi'l-maftûḥ min ʿilm al-ḥisâb

— Inṣâb al-khabr fî ḥisâb al-jabr

— Mîzân al-ʿulûm fî taḥqîq al-maʿlûm

— aț-Țafâḥafi aʿmâl al-misâḥa

See Matvievskaja, G. P., B. A. Rozenfel'd, *Matematiki i astronomy musul'manskogo srednevekov'ja i ich trudy (VIII-XVII vv)*, Moskova 1983, vol. 2, No 359, p. 381

[22] See concerning the relevant conjecture by Sezgin: Sezgin, F., *Geschichte des arabischen Schrifttums*, vol. 5, Mathematik, Leiden 1974, pp. 165f

[23] Ismâ'îl ibn Ibrâhîm ibn Fallûs, *op.cit.*, Ms 5970, Lbg. 199, for instance, f 25ª, 15-f 25ᵇ, 9

the author himself states, that there are no mathematical rules for their formation, are an example there of.[24] Nevertheless, even this kind of numbers seems to have been transmitted by other scholars too, since Ḥajjî Khalîfa makes reference to it in his encyclopaedia "Kashf aẓ-Ẓunûn":

" علم الخواص ..... ومنها خواص الاعداد المتحابة والمتباغضة كما بين في تذكرة الاحباب في بيان التحاب ".[25]

"The science of the special properties. ... Among these are the properties of the amicable and inimical numbers, as explained in 'Tadhkirat al-aḥbâb fî baiyân at-taḥâbb'."

The book, mentioned by Ḥajjî Khalîfa, could be the work of the same name by Kamâl ad-Dîn al-Fârisî.[26] In Al-Fârisî's treatise there is not the least trace of the inimical numbers in accordance with its orientation toward mathematical research. No other work, however, with such a title is known to me.

Ibn Fallûs' text consists of four parts — an introduction on the subject, the principles and the characteristics of the science of numbers, and three chapters about the classes of numbers, their names and rules for construction as well as 25 propositions, mainly from algebra and especially devoted to the solution of quadratic equations. The author calls these propositions geometrical principles and general theorems, which he took according to his introduction from mathematical books, according to his third chapter from geometrical books.[27] A first comparison of those theorems with the author's algebraical treatise[28] suggests that the algebraical works by al-Karajî and ʿUmar al-Khayyâm were the main sources of this third chapter. This part contains different variants for working out $(a+b)^2$, the computation of $(a+b)^3$, several further propositons, and as an added 26th theorem the recursional formula for the binomial coefficients: $\binom{n}{k} = \frac{n-(k-1)}{k}\binom{n}{k-1}$ in verbal form.[29]

[24] *ibid.*, f 21b, 14 - f 22a, 4.
[25] *Keşf-el-Zunun*, Kâtib Çelebi, Maarif Matbaası, İstanbul 1941, vol. 1, col. 725f.
[26] Compare footnote 13.
[27] Ismâîl ibn Ibrâhîm ibn Fallûs, *op. cit.*, f 16b, 6f.
[28] *ibid.*, f 25b, 11.
[29] *ibid.*, f 25b, 15 - f 26a, 3 and f 28a, 12 - f 28b, 4.

Aḥmad ibn as-Sirâj, who wrote one of the known three manuscripts, added to the text another problem, connected with the so called theorem of Wilson, which seems to have been formulated first by Ibn al-Haytham. Aḥmad ibn as-Sirâj gave the general solution of this problem in verbal form.[30]

The perfect and amicable numbers are sections 11 and 13 of chapter 2. The amicable numbers are divided into three kinds: amicable numbers according to quantity, amicable numbers according to quality, and amicable numbers according to quantity and quality. The text includes a table, which contains in general ten examples of nearly all kinds of numbers. There are no examples of amicable numbers of the second and third kind, inimical numbers, and three special types of solid numbers, the so-called board, brick, and well numbers.[31] The table gives only the first pair 220, 284 of the normal amicable numbers. In the column for perfect numbers there are ten examples,[32] seven of which are perfect numbers (disregarding scribal errors and minor mistakes in the calculation):6, 28, 496, 8128,[33] 33 550 336, 8 589 869 056,[34] 137 438 691 328. Thus, Ismâʿîl ibn Ibrâhîm ibn Fallûs' treatise contains the earliest recently known statement of the fifth, sixth, and seventh perfect number. At least half a century later (between 1292 and 1306) Quṭb ad-Dîn ash-Shîrâzî composed his encyclopaedia, in the mathematical part of which the fifth and sixth perfect numbers are found.[35] Ibn Fallûs' computation of perfect numbers

[30] *ibid.*, f 28^b, gloss.

[31] See for instance *Nicomachi Geraseni Pythagorei Introductionis Arithmeticae Libri II,* Rec. R. Hoche, Lipsiae, 1913, II, 17, 6.

[32] Ismâʿîl ibn Ibrâhîm ibn Fallûs, *op. cit.*, f 29^a, 17 f.

[33] The manuscript has 68128, obviously because the scribe of the manuscript wrote the last digit of the following number 1 130 816 twice. Aside from 1 130 816, the sixth (4096 128) and the tenth (35 184 367 894 538) number in the list are not perfect.

[34] The manuscript has 8 589 866 056.

[35] Muzafarova, Ch. R., "O matematičeskich glavach enciklopedičeskogo proizvedenija "Durra-at-tadž li gurra-at-dibadž" (Žemčužina korony dlja ukrašenija dibadža) Kutbaddina Širazi", *Učenie zapiski trudy mechaniko-matematičeskogo fakul'teta,* vol. 1, Dušanbe 1970, pp. 85-93, p. 92; The paper, however, contains two errors: First, the fifth perfect number is 33 550 336, not 33 550 366. Secondly, this number does not result for n = 16, but for n = 12. Since Quṭb ad-Dîn ash-Shîrâzî also gives the sixth perfect number, not mentioned by Muzafarova, she evidently confused the two perfect numbers, because the sixth evolves for n = 16.

corresponds essentially to Euclid's rule IX, 36.[36] The work "At-Takmila fi'l-ḥisâb" of Abû Manṣûr al-Baghdâdî (d 1037) contains an interesting statement about the perfect numbers. Its author refuses the antique assumption, that there is one perfect number in each degree of ten, and declares, that there is no such number between ten thousand and hundred thousand:

'' وقد غلط من قال : في كل عقد من العقود عدد واحد تام ، إذ ليس فيما بين عشرة آلاف ومئة ألف : عدد تام.'' [37]

"He, who said: in every tenary one number is perfect, is wrong, since there is no perfect number between ten thousand and hundred thousand."

It seems probably, that the fifth perfect number, at least, was known before Ibn Fallûs. Concerning the amicable numbers Ibn Fallus gives only for the first kind, i.e. the amicable numbers according to quantity, a construction rule. These amicable numbers of the first kind are the usual amicable numbers of other texts, defined by the condition that a_1, a_2 are amicable, if and only if $G_0(a_1) = a_2$, $G_0(a_2) = a_1$, where $G_0(n)$ indicates the sum of the proper divisors of n. This definition seems to have been exclusively used by the scholars of the Islamic Middle Ages. That it is equivalent to the assertion:

a_1, a_2 amicable numbers, if and only if $G(a_1) = G(a_2) = a_1 + a_2$, where G(n) denotes the sum of all divisors of n, was discovered ready by Thâbit ibn Qurra and stated at the end of his above mentioned treatise.[38]

[36] Ismâʿîl ibn Ibrâhîm ibn Fallûs, *op. cit.*, ff 20ᵇ - 21ᵃ.

[37] Abû Mansûr, Abd'l-Qahir ibn Tahir Al-Bâghdâdî, *op.cit.*, p. 227.

[38] '' وانه اذا اخذ كل جزء لكل واحد منهما وجمع ذلك كله مما كانت ذلك مثل ذينك العددين مجموعين.''

Saidan, A. S., *op. cit.*, p. 53

If every (divisor) of each of the two (numbers) is taken and all these (divisors) of each (of the numbers) are added, the sum of these (divisors) equals the sum of those two numbers.

See also "Sabit ibn Korra, Matematičeskie traktyty," Sostavitel B. A. Rozenfel'd, *Naučnoe nasledstvo*, vol. 8, Moskva 1984, p. 126

Ibn Fallûs expressed the rule for amicable numbers of the first kind in the following way:

" واما النوع الثالث عشر وهو الاعداد المتحابة فهى على ثلاثة اقسام متحابة في الكمية بان يكون احد العدد ين زائدا والاخر ناقصا ويكون اجزأ كل واحد منهما مساويةلكمية الاخر مثل ٢٢٠ و ٢٨٤ فان اجزأ كل واحد منهما مساوية لكمية الاخر وتوليدها من الاعداد الشطرنجية بان تجمعها فان حصل اول زد نا عليه اخرها ونقصنا منه ما قبل اخرها فيحصل اولان تضرب احدهما في الاخرثم المبطغ في اخر الاعداد يخرج اول المتحابين ثم تستخرج الثاني بزيادة احد الاولين على الاخر وتضرب المبلغ في اخر الاعداد فما حصل فهو الفضل عن المتحابين فتزيده على اول المتحابة يخرج العدد الثاني وعلى هذا توليد ها الى غير النهاية." [39]

"The thirteenth class, namely the amicable numbers, consists of three subkinds. (The first kind consists of) the amicable numbers according to quantity, (that is to say), that one of the two numbers is abundant and the other one is deficient. The (sum of the) parts of each of the two (numbers) equals the quantity of the other (number), like 220 and 284, because the (sum of the) parts of each of these equals the quantity of the other.They arise from the chess board numbers (in the following way:) We add them. If a prime number results, we add the last (of the numbers summed up) to (the sum), and we subtract from it the (number), which comes before the last of (the added numbers).

(If) two prime numbers result, we multiply them, and we multiply the result by the last of (added) numbers. Thus the first of the two amicable (numbers) results. Then the second is found by adding the first of the two prime numbers to the other one and by multiplying the result by the last of the (added) numbers. The (product) is the difference between the two amicable numbers. You add this to the first amicable number and the second amicable number is the result. In this way they are constructed (in) unlimited (number)."

The above mentioned chess board numbers are the so called even times even numbers,[40] i.e. the powers of two. The procedure described by Ibn Fallûs is thus the following:

[39] Ismâ^{c}îl ibn Ibrâhîm b. Fallûs, *op. cit.*, f 21^a, 14-21^b, 8.

[40] *ibid.*, f 18^b, 7 f.

If $\sum_{0}^{n} 2^k = 2^{n+1} - 1$ is prime, form

$$p_1 := \sum_{0}^{n} 2^k + 2^n$$

and $p_2 := \sum_{0}^{n} 2^k - 2^{n-1}$.

If p_1, p_2 are primes, the first amicable number is $a_1 := p_1 \cdot p_2 \cdot 2^n$. The second amicable number arises, when the difference $(p_1 + p_2)2^n$ between the both amicable numbers is added to a_1 : $a_2 := (p_1 + p_2)2^n + a_1$.

This rule corresponds according to R. Rashed[41] to that given by Ibn Sînâ in his encyclopaedia "Kitâb ash-Shifâ", after the edited text of ᶜA. L. Maẓhar has been corrected.[42] Rashed, however, did not explain, what correction he had in mind. In my opinion, the text is clear as is and is not in need of any correction, after the meaning of the dual suffixed personal pronoun humâ in the beginning of the passage has been clarified. Ibn Sînâ surely knew, that $2^{n+1} - 1$ is not always prime. Thus, the use of the dual (humâ) instead of the plural (hâ) is imperative and means, that Ibn Sînâ referred to the example 220, 284 mentioned in his text just before. In this case is $n = 2$ and therefore the first two even times even numbers 2 and 4 are to be added to 1. The result happens to be a prime. Thus, R. Rashed's interpretation has to be modified in the sense, that Ibn Sînâ did not state the general condition of $2^{n+1} - 1$, $n = 1, 2, \ldots$, being prime, but only the primality of p_1 and p_2. The further form of Ibn Sînâ's rule agrees not only in the contents, but also in the generality of the wording with the rule of Ibn Fallûs, but Ibn Fallûs does not repeat the restriction to the case $n = 2$ in the first part of the rule. If in Ibn Sînâ's expression something is to be corrected at all,one should avoid the second dual suffixed personal pronoun, i.e. instead of ᶜalaihimâ one should

[41] Rashed, R., "Nombres amiables, parties aliquotes et nombres figurés aux XIII^{eme} et XIV^{eme} siècles," *op. cit.*, p. 116, footnote 30^a.

[42] Rashed writes: Si l'on corrige la lecture de l'édition, le texte d'Avicenne devient lumineux, et se traduit ainsi:

si $(2^{n+1} - 1)$, p_{n-1}, p_n sont premiers, alors $2^n p_{n-1} p_n$ et $2^n(p_{n-1} + p_n + p_{n-1}p_n) = 2^n q_n$ sont amiables.

Here his abreviations mean:

$p_{n-1} = 3 \cdot 2^{n-1} - 1$, $p_n = 3 \cdot 2^n - 1$, $q_n = 9 \cdot 2^{2n-1} - 1$. Compare ibid., pp. 111 and 116, footnote 30^a.

read ʿalaihâ or alaihi. This correction, however, affects the meaning of the text only in a minor way.

Ibn Sînâ wrote:

'' وإذا جمعت أعداد زوج الزوج والواحد معهما فاجتمع عدد أول بشرط أن يكون اذا
زيد عليهما آخرها ونقص الذى قبله كان المبلغ بعد الزيادة والمبلغ بعد النقصان أولين [43]
فضرب المبلغ المزيد عليه في المبلغ المنقوص ثم ضرب ما اجتمع في آخر المجموعات حصل
عدد له حبيب ، وحبيبه العدد الذى يكون من زيادة مجموع الزائد والناقص المذكورين ضربا
في آخر المجموعات علي العدد الموجود أولا الذى له حبيب وهما متحابان.'' [44]

"If two even numbers are added together and to the sum (number) one, a prime number results. If the last (of the added even times even numbers) is added to the two (summands, i.e., 6 + 1) (or better: to them, i. e., 2 + 4 + 1, or: to it, i. e., the prime number 7) and the number before it is subtracted (and) if then the (sum) and the (difference) are two prime numbers, then the product of the (sum), the (difference) and the last of the summed (numbers) gives a number, which has a friend. Its friend is the number, which arises from the addition of the sum of the mentioned sum and difference, multiplied with the last of the summed (numbers), to the previously found number which has a friend. These two are amicable (numbers)."

This reading and interpretation of Ibn Sînâ's rule lets unexplained how Ibn Fallûs' general primality condition for $2^{n+1} - 1$ could have been evolved. A text, which contains an argument for the choice of $2^{n+1} - 1$ instead of Thâbit ibn Qurra's $2^{n+1}(2^{n+1} + 2^{n-2}) - 1$ is not known to me. There are some clues, however, out of which this step could be reconstructed hypothetically. First, a check of Ibn Sînâ's rule for $n = 3$ already shows, that, although p_1 and p_2 are primes, a_1 and a_2 are not amicable numbers and that, in contrast to $n = 2$, $2^4 - 1 = 15$ is not prime. Secondly, the continuation to the the case $n = 4$ yields p_1, p_2, and $2^5 - 1 = 31$ as prime numbers, and a_1 and a_2 as amicable ones. According to the convincing interpretation of Thâbit ibn Quarra's treatise on amicable numbers by J. Hogendijk, Thâbit ibn Qurra already knew this pair of

[43] Here, the edition's awwalîyan has been changed to awwalain.

[44] Ibn Sînâ, *op. cit.*, p. 28, 15-20.

amicable numbers, because the proof of his general rule is phrased for the case $n = 4$.[45] Hence it could have been possible to derive the rule cited by Ibn Fallûs heuristically from the rule of Ibn Sînâ.

Thirdly, the writings of other authors show that Thâbit ibn Qurra's rule was not transmitted verbatim. Nevertheless, the majority of the known texts follow it essentially.[46] The only known scholar, who lived be-

[45] Hogendijk, J. P., "Thâbit ibn Qurra and the Pair of Amicable Numbers 17296, 18416," *Historia Mathematica*, vol. 12, 1985, pp. 269-273.

[46] al-Qabîsî uses the Thâbit-rule in the form given by Thâbit Ibn Qurra: If $p_1 = 2^{n+1} - 1 + 2^n$, $p_2 = 2^{n+1} - 1 - 2^{n-1}$ and $p_3 = 2^{n+1}(2^{n+1} + 2^{n-2}) - 1$ are prime numbers, then $2^n p_1 p_2$ and $2^n p_3$ are amicable numbers, although one sentence is missing in the only extant manuscript. Risâlat al-Qabîṣî fi jamᶜ anwâᶜ min al-aᶜdâd, *op. cit.*, pp. 192-191 (in Arabic);

al-Karajî summarizes Thâbit ibn Qurra's text in a relatively extensive manner in a chapter of his algebraical treatise "al-Badîᶜ" without reference to his source. He uses in it the following version of Thabit's rule for amicable numbers:

$a_1 = p_1 p_2 2^n$, $a_2 = ((2^{n+1})^2 + \frac{1}{8}(2^{n+1})^2 - 1)\, 2^n$, p_1, p_2 as defined by Thabit ibn Qurra and prime numbers, $((2^{n+1})^2 + \frac{1}{8}(2^{n+1})^2 - 1)$ prime number. "L'Algèbre al-Badîᶜ d'Al-Karagî," *op. cit.*, p. 27

Abû Manṣûr al-Baghdâdî introduced in the rule, basicly similar to Ibn Sînâ's version, a recursive description of the prime numbers p_1 and p_2:

— first step: $2^2 + 1 = 5$ is a prime number,

$5.2 + 1 = 11$ is a prime number,

$11 - 2^2$ is a prime number, i. e., he tests, if $2^{n+1} - 1$ is prime;

then $11 \cdot 5 \cdot 2^2 = 220$ is the first amicable number and $(5 + 11) \cdot 2^2 + 220 = 284$ is the second amicable number;

— second step: search for a new pair of amicable numbers $2 \cdot 11 + 1$, $2 \cdot 5 + 1$, $2 \cdot 11 + 1 - 2^3$, $2 \cdot 5 + 1 - 2^3$. Test, if these four numbers are primes, if not, this step does not yield a new pair and computate again:

$2p_2 + 1$, $2p_1 + 1$ and test, if they are primes. Now the condition, $2^{n+1} - 1$ and $2^{n-1} - 1$ prime numbers, is missing.

If in the n-th step $2p_2 + 1$ and $2p_1 + 1$ are prime numbers, then the first amicable number is $(2p_2 + 1)(2p_1 + 1) \cdot 2^n$;

Abû Mansûr, Abd'l-Qahir ibn Tahir Al-Baghdâdî, *op. cit.*, pp. 230 - 231

Al-Fârisî describes a version of Thâbit ibn Qurra's rule:

$q_1 = 2^n + 2^{n-1} - 1$, $q_2 = 3 \cdot 2^n - 1$, $q_1 q_2 = q_3$, $q_1 + q_2 + q_3 = q_4$; if q_1, q_2 and q_4 are prime, $2^n q_3$ and $2^n q_4$ are amicable numbers.

See Rashed, R., "Matériaux Pour l'Histoire des Nombres Amiables et de l'Analyse Combinatoire," *op.cit.*, p. 265; Zain ad-Dîn at-Tanûkhî (1307) states the rule as al-Karajî, the text, however, omits in the rule's general statement the condition for p_2: p_1, p_2 are defined as by Thâbit ibn Qurra, p_3 as by al-Karajî. If the three numbers are prime, then fore Ibn Fallûs and stated the rule for amicable numbers in a form,similar to that of Ibn Sînâ, is Abû Mansûr al-Baghdâdî.[47] Fourthly, the version of Thâbit ibn Qurra's text in the Aya Sofya manuscript, which differs from the Paris manuscript, offers a further clue.[48] In the Aya Sofya manuscript it is required that not only

p_1, p_2 and $p_3 := 2^{n+1}(2^{n+1} + 2^{n-2}) - 1$, but also a number $Z = 2^{n+1} - 1$ is prime.[49] On the basis of the above mentioned interpretation by Hogendijk this inclusion of Z in the primality condition for p_1 and p_2 also could be explained as a reflection of the concrete case $n = 4$, where $Z = 31$. Since the primality condition for p_1 and p_2 by Thâbit ibn

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$2^n p_1 p_2$ and $2^n p_3$ are amicable numbers. The absent p_2 can be derived from the following example 220, 284. Compare ibid., p. 228; Muḥammad Baqîr al-Yazdî phrases the rule as follows: $q_1 = \frac{3}{2} \cdot 2^n - 1 = 2^n + 2^{n-1} - 1$, $q_2 = 3.2^n - 1 = 2^n + 2^{n+1} - 1$, $q_3 = q_1 q_2$, $q_4 = q_1 + q_2 + q_3$.

Note, that al-Yazdî's general statement, as edited by Rashed, contains an error, because he defines q_1 as $2^n - 1$. But he gives q_1 as above in the example. His general rule continues: If q_1, q_2, and q_4 are prime numbers, then $2^n q_3$ and $2^n (q_1 + q_2 + q_3)$ are amicable numbers. Compare ibid., p. 226;

Ibn Haidur (d 1413) gives the rule in his commentary on the "Talkhîs a'mâl al-ḥisâb" by Ibn al-Bannâ in connection with the example 220, 284:

p_1, p_2 are defined as by Thâbit b. Qurra, if they are prime, then $2^n p_1 p_2$ is the first of the sought numbers. If $p_3 = (2^{n+1} + \frac{1}{4}. 2^n) . 2^{n+1} - 1$ is prime, then $p_3 2^n$ is the second one. See ibid., p. 217; In a treatise, attributed by M. Soussi to Ibn al-Bannâᶜ, but probably composed by a commentator according to Rashed, the rule again appears in the original form of Thabit ibn Qurra. The author of the text, however, erroneously requires the primality of the multiplicand $p_1 p_2$ of 2^n instead of each of the numbers p_1 and p_2. Then he states the contradiction between his requirement and the examples 220, 284 and 17296, 18416, without recognizing his mistake. See Soussi, M., *op.cit.*, pp. 4-7; Soussi's formulas of Thâbit's rule contain two inaccuracies (p. 13): instead of $c = 9 \cdot 2^{n-1} - 1$ read $c = 9 \cdot 2^{2n-1} - 1$. Thâbit's rule is not simpler than the rule given by this text, but the two rules are equivalent. This fact has already been pointed out by Borho. See Borho, W., "Befreundete Zahlen. Ein zweitausend Jahre altes Thema aus der elementaren Zahlentheorie," *Lebendige Zahlen. Mathematische Miniaturen I.* Basel, 1981, p. 30; Quṭb ad-Dîn ash-Shîrâzî, on the other hand, transmits the rule in the same form as Ibn Fallûs. See Muzafarova, Ch. R., "Arifmetika Nikomacha v izlozenii Kutbaddina Širazi," *op. cit.*, p. 129

[47] See the description of al-Baghdâdî's rule in footnote 46.

[48] Ms 4830, Aya Sofya, Istanbul, ff 110ᵃ - 121ᵇ; See Saidan, A. S., *op. cit.*, pp. 50 - 53.

[49] *ibid.*, p. 50.

Qurra points to the general case,[50] the inclusion of Z tends to become a general condition too. Later authors, who transmitted Ibn Sînâ's rule, could have taken over this general primality condition for $2^{n+1} - 1$, p_1, and p_2 on the basis of the version of Thâbit ibn Qurra's treatise in the Aya Sofya manuscript, while the condition for p_3 was omitted, because p_3 was no longer computated explicitely. They may also have introduced themselves the general primality condition for $2^{n+1} - 1$ by considerations as outlined above.

No Arabic-Islamic scholar seems to have investigated the relations between the both rules. A comparison between the pairs of amicable numbers, produced by the two rules, shows, that both yield the pairs for $n = 2$ and $n = 4$, but only Thâbit ibn Qurra's rule gives the pair for $n = 7$ too. According to Borho[51] this rule does not produce any other pair for $n \geqslant 20.000$. With the notation $q = 2^{n+1} - 1$ and p_3 as above as $2^{n+1}(2^{n+1} + 2^{n-2}) - 1$ the relation between the two rules can be stated as:

$$p_3 = \frac{9}{8}(q + 1)^2 = \frac{1}{8}(9q^2 + 18q + 1).$$

[50] ,, فان كان كل واحد من اعداد ز ، ح ، ط (١٠) عددا اولا غير الاثنين
فهو الذى نريد ، والا تجاوزنا الاعداد التي جمعت الي غيرها حتى ننتهى الى ما تكون هذه الاعداد
منه اوايل (١١١) .
,, (١٠) في النسخة ٢ : من عد د ى ح ، ط.
(١١) في النسجة ٢ : هذان العددان من عدد ين اولين.،،

ibid., pp. 50 and 54

For the English version of this passage I use the translation of Hogendijk, J., "Thâbit ibn Qurra and the Pair of Amicable Numbers 17296, 18416, *op. cit.*, p. 270. I replaced, however, Hogendijk's interpolation (*) through the original letter Z of the Arabic text, omitted Hogendijk's explanation [...] and added the numbers of Saidan's footnotes (10) and (11): "If each of the numbers Z, Ḥ, Ṭ[(10)] is a prime number other than the number two, then this is what we want. If not, then we proceed with the (series of) numbers that we added until we arrive at some number such that these numbers which are derived from it are prime.[(111 sic)]"

"(10) in manuscript 2: ... the two numbers Ḥ, Ṭ (11) in manuscript 2: ... these two numbers are prime."

The manuscript 2 is the Paris manuscript Ms 2457, Bibliothèque Nationale, ff 170[b] - 180[b]; Compare Sabit ibn Korra, *op. cit.*, p. 124.

[51] Borho, W., *op. cit.*, p. 14.

Evidently, the assertions "q prime" and "p_3 prime" are independent, as is demonstrated by the following examples: $n = 6$ $q = 127$ prime, $p_3 = 18.431 = 7.2633$ $n = 8$ $q = 511 = 7.73$, $p_3 = 294.911$ prime.

No quadratic polynom f is known which yields for infinitely many argument values x prime numbers f(x).[52] The analysis, which numbers $a_1 = 2^n p_1 p_2$ and $a_2 = 2^n p_3$ with p_1, p_2, and p_3 primes are amicable numbers, gives the following results:

— if a_1 and a_2 are amicable numbers, $G(a_1) = G(a_2)$ is valid for primes p_3 with $p_3 = (p_1 + 1)(p_2 + 1) - 1$ and arbitrary primes p_1 and p_2;

— if a_1 and a_2 are amicable numbers, $G(a_1) = G(a_2)$ holds true, be $2^{n+1} - 1$ a prime number or not, i.e. the primality condition for $2^{n+1} - 1$ is insignificant;

— the structure of the primes p_1 and p_2 of two amicable numbers a_1 and a_2 depends upon the equation $2^n p_1 p_2 + 2^n p_3 = G(2^n p_3$, which has to be valid for two amicable numbers a_1 and a_2.

Using the above given relation between p_3 and p_1, p_2 the equation $2^n p_1 p_2 + 2^n p_3 = G(2^n p_3)$ yields

$$\frac{(p_1 + 1)(p_2 + 1)}{p_1 + p_2 + 2} = 2^n.$$

If $a_1 = 2^n p_1 p_2$ and $a_2 = 2^n p_3$ are amicable numbers, the least of the three primes is greater than 2. Thus, we can put
$p_1 + 1 = 2k_1$, $p_2 + 1 = 2k_2$, where $k_1 < k_2$.

We get

$$\frac{4k_1 k_2}{2(k_1 + k_2)} = 2^n.$$

With $k_1 = 2^{n-1-k}(2^k + 1)$

and $k_2 = 2^{n-1}(2^k + 1)$ and $O < k < n$
one obtains the following structure of the three primes:

[52] I am indebted to Dr. O. Neumann, Jena, for this information.

$$p_1 = 2^{n-k}(2^k + 1) - 1$$
$$p_2 = 2^n (2^k + 1) - 1$$

$$p_3 = (p_1 + 1)(p_2 + 1) - 1 = 2^{2n-k}(2^k + 1)^2 - 1.$$

This results is equivalent to Euler's rule for all amicable numbers of the above given type, i.e., $a_1 = 2^n p_1 p_2$ and $a_2 = 2^n p_3$. For $k = 1$ one gets an equivalent version of Thābit b. Qurra's rule.

Ibn Fallûs describes the second kind of amicable numbers as follows:

" المتحابة في الكيفية بان يكون احد العدد ين زوجا ويكون اجزاؤه فردا ويكون الاخر فردا ويكون اجزاؤه زوجا." [53]

"(Numbers) are amicable according to quality, if one of the two numbers is even and its parts are odd, and if the other (number) is odd and its parts are even."

This statement is meaningful, provided that the two words "its parts" are interpreted as "sum of its parts", i.e., sum of the proper divisors of the number. This interpretation is justified by the wording of the definitions for abundant, deficient and amicable numbers according to quantity in the second chapter of Ibn Fallûs'treatise. There the author uses the expression "parts" in the sense of "sum of the parts", in contrast to the definitions of the first chapter.[54] Hence two numbers are called amicable according to quality, if

$a_1 = 2k_1$ and $G_0(a_1) = 2m_1 + 1$,

$a_2 = 2k_2 + 1$ and $G_0(a_2) = 2m_2$,

$k_i, m_i \in \mathbb{N}, i = 1, 2.$

Examples are not given in the text. It is, however, easy to show, that only the following numbers are amicable according to quality:

a_1: $2^n, (2k_1)^2, 2^n(2k_1)^2, 2^n(2k_1 + 1)^2$

[53] Ismâ'îl ibn Ibrâhîm ibn Fallûs, *op. cit.*, f 21^b, 9f.

[54] *ibid.*, f 17^b, 1-5.

$a_2: (2k_2 + 1)^2$,

$n, k_1, k_2 \in \mathbb{N}$.

$(2k_1)^2$ and $2^n (2k_1)^2$ are nothing but special cases of 2^n or $2^n (2k_1 + 1)^2$. $a_1 = 2^n$ obviously satisfies the conditions. $a_1 = 2_n (2k_1 + 1)^2$ and $a_2 = (2k_2 + 1)^2$ satisfy them, if and only if $G[(2k_1 + 1)^2]$ and $G[(2k_2+1)^2]$ are odd. Let $2k+1 = p_1^{n} \cdot p_2^{r2} \cdot \ldots \cdot p_t^{n}$ be the prime number decomposition with $p_i \neq p_j \neq 2$, $\forall\, i, j \in \{1, \ldots, t\}$.

$$G[(2k+1)^2] = \prod_{i=1}^{t} \left(\sum_{l=0}^{2ri} p_i^{l} \right)$$ is odd, if and only if every factor is odd.

This is evident, because every factor is the sum of an odd number of odd terms. It follows, that all other even or odd numbers are not amicable according to quality.

The third kind, the amicable numbers according to quantity and quality, is defined by Ibn Fallûs as those numbers, which possess the properties of the both preceding kinds.[55] This definition means:

Two numbers a_1, a_2 are called amicable according to quantity and quality, if a_1 is even, $G_0(a_1)$ is odd, a_2 is odd and $G_0(a_2)$ is even with $G_0(a_1) = a_2$ and $G_0(a_2) = a_1$. Thus, one asks implicitely for pairs a_1, a_2 of opposite parity with

1. $a_1 = 2^n$, $a_2 = (2k_2 + 1)^2$

or

2. $a_1 = 2^n(2k_1 + 1)^2$, $a_2 = (2k_2 + 1)^2$

and $G_0(a_1) = a_2$, $G_0(a_2) = a_1$.

The first case is simple to exclude, since Kanold has proven, that two numbers $s = p^n$, $t = q_1^{m_1} \cdot q_2^{m_2} \ldots q_r^{m_r}$, $p \neq q_i \neq q_j$ can only be amicable, if s, t, n and some m_i are odd[56]. From his proof it follows in-

[55] *ibid.*, f 21b, 10-12.

[56] Kanold, H.-J., "Über befreundete Zahlen. I," *Mathematische Nachrichten*, vol. 9, 1953, pp. 243-248.

cidentally, that $2k_2 + 1$ in the second case can not be a power of a prime number. Hence, a pair of numbers with opposite parity can be amicable (this formulation is equivalent to Ibn Fallûs' amicable according to quantity and quality) only, if $a_1 = 2^n(2k_1 + 1)^2$, $a_2 = (2k_2 + 1)^2$ and $2k_2 + 1$ is not a power of prime number.[57] Such a pair is unknown until present,[58] but necessary conditions and lower bounds have been derived:

Let $a_1 = 2^n(2k_1 + 1)^2$ and $a_2 = (2k_2 + 1)^2$ be amicable numbers. Then the following is valid:

1. a_1 is neither a fourth power nor a quadruple or octuple of such a power.

2. $2k_1 + 1$ is not a square.

3. If $n = 1$, we have $a_2 < a_1$, $(a_1, a_2) = 1$; a_2 possesses at least five distinct prime factors, $a_1 a_2 \equiv 2 (\mathrm{mod}\ 24)$, $a_2 > 10^{60}$.

4. If $n > 1$, then $a_1 < a_2$, $(a_1, a_2) > 1$.

5. If $n > 1$ odd, then $(2k_1 + 1, 3) = (2k_2 + 1, 3)$, and there is a prime number q and $m \in \mathbb{N}$ with $q^m | 2k_2 + 1$ and $q^{m+1} \nmid 2k_2 + 1$ and $q \equiv m \equiv 1 \ (\mathrm{mod}\ 3)$. If $m \equiv 3 (\mathrm{mod}\ 4)$, there exists a prime number p, not necessarily distinct of q, and $1 \in \mathbb{N}$ with $p^1 \mid 2k_2 + 1$ and $p^{1+1} \nmid 2k_2 + 1$ and $2p \equiv 1 \equiv 2 (\mathrm{mod}\ 5)$ and

[57] After the completion of the manuscript I was able to consult the treatises Kanold, H.-J., *op. cit.*, and Gioia, A. A., A. M. Vaidya, "Amicable Numbers with Opposite Parity," *American Mathematical Monthly* 74, vol. 8, 1967, p.p 969-973, cited by Lee, E. J., J. S. Madachy, "The History and Discovery of Amicable Numbers"- Part I, *Journal of Recreational Mathematics 5*, vol. 2, 1972, pp. 77-93. The survey of Lee and Madachy served as the basis for the considerations in the present paper about the structure of amicable numbers with different parity.The treatise of Gioia and Vaidya also proves, that amicable pairs of different parity can only have the structure $2^n(2k_1 + 1)^2$, $(2k_2 + 1)^2$, $2k_2 + 1$ being a composed number. Some further propositions on the structure of $2k_1 + 1$ are also derived. The proof is independent of Kanold's paper, which gives a sharper result concerning the structure of $2k_2 + 1$. Note that Gioia's and Vaidya's paper operates with an incorrect statement of the definition of amicable numbers. They require, that the sum of all positive divisors of one number is equal to the other number (p. 69).

[58] According to Lee, E. J., J. S. Madachy, *op. cit.* p. 84, the formula $a_1 = 2^nM^2$, $a_2 = N^2$, M, N odd numbers, was first given by Gmelin, O., *Über vollkommene und befreundete Zahlen, Diss.*, Hiedelberg, 1917, Halle/S., 1917, but as a matter of fact its earliest occurence is in: Kanold, H.-J., "Über befreundete Zahlen. II," *Mathematische Nachrichten*, vol. 10, 1953, pp. 99 - 111, p. 99.

$$2k_1 + 1 \equiv 2k_2 + 1 \equiv \frac{1}{4}(n + 1)\, G[(2k_1 + 1)^2] \equiv 0 \pmod 5.$$

6. If $2k_1 + 1 = p^s$, then $n = 1$, $s > 6$, $p \equiv 1 \pmod{12}$, the number of distinct prime factors of a_2 is greater than 24 and $a_2 > 10^{75}$.[59]

These results obtained by 20th century mathematicians show that the problem of finding amicable numbers of the third kind could not be solved by medieval mathematicians. Thus this problem probably did not originate in a mathematical context. Since no other text is known to contain this problem, an ultimate answer about its origin or its motivation cannot be given at this time. Ismail ibn Ibrahîm ibn Fallûs' way of expression, however, points at its possible origin in a philosophical background.

59 *ibid.*, pp. 99 - 111; Lee, E. J., J. S. Madachy, *op. cit.*, p. 84.

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Dr. Sonja Brentjes

Karl-Sudhoff-Institut für Geschicte der Medizin und Naturwissenschaften,

Bereich Medizin, Karl-Marx-Universität

Talstr. 33, Leipzig, DDR-7010.