# TRIGONOMETRY IN THE SIXTEENTH CENTURY COPERNICUS AND TAQÎ AL DÎN

SEVİM TEKELİ *

The works on trigonometry, that is to say "Triangle Measurement", go back to the Egyptians and the Babylonians.

In Greece, Autolycos of Pitane[1] (forth century B. C.), Aristarchos of Samos[2] (third century B. C.), Hiparchos[3] (second century B.C.) and Heron of Alexandria (first century A.D.) are the forrunners of trigonometry. The *Spherics* of Menalaos of Alexandria (first century A.D.) are, in fact a treatise on spherical trigonometry. As we come to Ptolemaus (second century A.D.), like other Greek writers, he used chords and extended the table of chord.

In the West when the period of regression began, the Hindus produced Siddhantas appearently based on the Greek works. The *Surya Siddhânta* of Surya (tenth century A.D.) is the only one which seems to be completely extant. The most important feature of this treatise is the use of sines, instead of chords and versed sines, The *Paulisa Siddhânta* is equally important in the History of Trigonometry.[4]

Later in Islâm, Al-Battânî al-Ṣâbiî (858–929 A.D.), the greatest astronomer of his time, used sines regularly with a clear consciousness of their superiorty over the Greek chords. He also introduced tangents

* Professor of the History of Science, Head of the Department of Philosophy and the Chair of the History of Science, Faculty of Letters, Ankara University.

[1] George Sarton, *Introduction to the History of Science*. Washington 1927, vol. 1, p. 141-142.

[2] The next important step in the development of the trigonometry was taking by Aristarchus. He attempted to find the distances from the earth to the sun and the moon, and the diameters of these bodies. David Eugene Smith, *History of Mathematics* vol. 2. New York 1958. p. 604.

[3] Sarton, 1 vol. p. 193-194.

[4] The most important feature of this comprehensive treatise is the use of sines (jyâ) instead of chords. (Sarton, vol. 1. p. 387). But in these works they speak of the half chord in place of chords as it is known to define the sines this way can not help to point out the other trigonometric functions.

cotangents. He made a special study of tangents, calculating a table of tangents, introduced the secant and cosecant.[5]

In the thirteenth century, Naṣîr al-Dîn al-Ṭûsî wrote the first book, *Shakl al-Qaṭṭâ*, in which trigonometry appeared as a science by itself.[6] The important relation, now expressed "*The theorem of sines*" allthough recognized by Al-Beyrûnî (eleventh century A.D.) and Abû'l-Wafâ, it was Naṣîr al-Dîn al-Ṭûsî who first expressed it clearly.[7]

Muslim Scolars such as Al-Beyrûnî[8] and Abû'l-Wafâ[9] developed new ways to compute the sin 1°. In the fifteenth century Ghiyâth al-Dîn al-Kashî solved this problem by using an original method, that is to say, producing a cubic equation. Qâdîzâde-i Rûmî[10] (fifteenth century), Uluğ Bey (fifteenth century) were also occupied with this problem but their methods were not new.

The astronomers, Al-Zarqâlî[11] (eleventh century), who constructed the trigonometrical tables, and Jâbir ibn Aflah[12] (tweefth century) must also be added to this list particulary relating to the spherical triangles in Spain.

Coming to the Western World, in thirteenth century Fibonacci was familiar with the trigonometry of Muslims in his *Practica Geometriae*.[13]

[5] Born before 858 in or near Harrân. Flourished at Raqqa, and died in 929 near Sâmarrâ. He wrote various scientific books. His main work is *De numeris stellarum et motibus* translated in to Latin in 12th century by Robert of Chester and by Plato of Tivoli. A century later, by the order of Alphonso X it was translated into Spanish. Plato's translation was published in 1537 in Nürnberg. It was extreemly influential untill the Renaissance. Sarton. vol. 1. p. 602-603.

[6] Smith, vol. 2. p. 609.

[7] Smith, vol. 2, p. 630.

[8] In finding the sin 1°, Beyrûnî followed the way of the trisection of an angle of 3° in his *Qânûn al-Mas'ûdî*. Carl Schoy, "Beiträge zur Arabischen Trigonometrie", p. 365-399.

[9] The method of Abû'l-Wafâ is given by Salih Zeki. *Asâr-i Bâkiye*. vol. 2. İstanbul 1329 H., p. 106-120.

[10] Salih Zeki, p. 133-139

[11] Smith, p. 609.

[12] Smith, p. 609.

[13] Smith, p. 609.

Although in the forteenth century English Scolars such as Maudith,[14] Richard Wallingford[15] and Jean de Linieres[16] knew Muslim trigonometry. Peurbach (1423–1461) and Regiomontanus were well acquinted with it. Regiomontanus' *De Triangulis omnimodis Libri V* (written 1464) first published in Nurnberg 1533, established trigonometry as science independent of astronomy. He computed new tables and laid the foundation for later works on plane and spherical trigonometry. Thank A. von Braunmühl for his 'Nassîr Eddîn Tûsi und Regiomontan.' *(Abhandlungen der Kaiserlich Leopoldinisch - Carolinischen Deutschen Akademie der Naturforscher, LXXI, (1897).*[17]

In the sixteenth century, as David Eugene Smith says in his *History of Mathematics.* vol II (Dover Publication) p. 610 'Copernicus completed some of the work left unfinished by Regiomontanus in his famous book, *De Revolutionibus Orbium Coelestium.* Later the chapter, *De Lateribus et Angulis Triangulorum* of this book, published separetly by his pupil Rhaeticus.[18]

On the other hand, that is to say, in the Islamic World, as well as Ottoman Empire, there is still no knowledge on the history of trigonometry in this Century and the following Centuries.

The purpose of my work is to present this subject depending the *Sidra al-Muntahâ* of Taqî al-Dîn, the famous astronomer and the mathematicien of his time, the founder of the Istanbul Observatory in 1575.[19]

[14] English mathematician and astronomer. One of the founders of Western trigonometry. Sarton, vol. 3, part 1, p. 660-661.

[15] The greatest English mathematician of his time, one of the introducers of trigonometry into Christian Western Europe. The *Quadripartitum de sinibus demonstratis* is the first original Latin treatise on trigonometry. Sarton, vol. 3, part 1, p. 664-668.

[16] His second *Canones* is on trigonometry. John gave tables of sines for each half-degree, and tables of tangents for every degree. These tables are of Arabic origin. Sarton, vol. 3, part 1, p. 649-652

[17] Smith, p. 609-610.

[18] Smith, p. 610.

[19] Taqî al Dîn Muhammad al Rashîd ibn Ma'rûf born in Egypt in 1526, educated there. In 1575, built one of the most important observatories in Islâm, in Istanbul. Sevim Tekeli "Nasirüddin, Takiyüddin ve Tycho Brahe'nin Rasat Aletlerinin Mukayesesi". *A. Ü. Dil ve Tarih-Coğrafya Fakültesi Dergisi.* vol. 16. No. 3-4 (1958), p. 301-393.

The best and shortest way to show what Taqî al-Dîn did relating to trigonometry is to make a comparison between the chapter, *De Lateribus et Angulis Triangulorum* of the *De Revolutionibus Orbiom Coelestium* of Copernicus, and the third chapter of *Sidra al-Muntahâ*[20] of Taqî al-Dîn.

*The straight lines in a circle:*

The twelfth section of the first book of the *De Revolutionibus Orbium Coelestium* of Copernicus and the first section of the first book of the *Sidra al-Muntahâ* of Taqî al-Dîn are on the straigth lines in a circle. Copernicus divides the circle into 360° and the diameter into 2000000 parts. On the other hand Taqî al-Dîn divides the circle into 360° and the diameter into 120 parts, I.e., each part being 2. In the fifteenth century, Regiomontanus divided the diameter into 10000000 parts. Before the invention of decimal franction, adopting the diameter $120^p$, $2000000^p$, $10000000^p$ are very useful.[21]

On the other hand, in the West, the first to adopt the simpler form

$$\sin 90° = 1$$

was Just Bürgi in the seventeenth century. This is an important point in Taqî al-Dîn's favour.[22] The first geometrician who divided the diameter into 2 parts was Abû'l-Wafâ in the tenth century.[23]

Both to set forth a table of chords have used the formulaes known by Ptolemy as ch 2A, ch A/2, ch (A–B), ch (A + B). Up to this point, there is a parallelisme between Copernicus and Taqî al-Dîn.

*Chord 1° or 2°:*

It is evident that by using these formulaes chords subtending 3°,1/2°, 3/4° can be calculated, but some chords as ch1°, or ch2° will be skipped.

Copernicus calculated ch 1° or ch 2° by proving the arc is always greater than the straight line subtending it, but in going from greater to lesser sections of the circle, the inequality approches equality.[24]

[20] As it has not yet been published one of the manuscript copies is used

[21] J. D. Bond "The Development of Trigonometrical Method down to the Close of the Fifteenth Century". *Isis*, vol. 4. 1922, p. 304.

[22] Smith, p. 627.

[23] Bond, p. 302.

[24] Copernicus, p. 537.

As we come to Taqî al-Dîn, he says the fallowing, "The Ancients could not find a correct way to get the ch 1° or ch 2°, in consequence of this, they depended on approximate methods which is not worth to describe. The Late Ulug Beg said 'We had inspiration about extracting ch 1° and sin 1°'.[25] This method involves the approximate solution of a cubic equation of the form[26]

$$ax-b = x^3$$

*The works on the sines:*

At the end of the twelfth section, Copernicus, without using the term of sines, says, "Neverthless I think it will be enough if in the table we give only the halves of the chords subtending twice the arc, whereby we may concisely comprehend in the quadrant what it used to be necessary to spread out over the semicircle; and especially because the halves come more frequently into use in demonstration and calculation than the whole chords do."[27] The knowledge given by Copernicus related to the sines consists of this.

As we come to Taqî al-Dîn, the 2th section of the book I of the *Sidra al-Muntahâ* is on the sines. In this section, he gives the definitions of sine, cosine, secant, and the formulaes of sin (A–B), sin (A + B), sin 2A, sin A/2[28], and calculates sin 1°.[29]

As it is seen Taqî al-Dîn says everything on sine, cosine, secant, on the contrary Copernicus mentions only the halves of the chords subtending twice the arc, anyhow the halves of the chord subtending twice the arc are equal to the sines of this arc. To define the sine this way can not help to point out the other trigonometric functions as cosine, secant and cosecant. So the halves of the chord are not considered an important step in the history of trigonometry.[30]

*On the plane triangles:*

The thirteenth section of the *De Revolutionibus* is on the plane triangles. He prooves that,

[25] Sevim Tekeli, "Taqî al Dîn's works on Extracting the Chord 2° and Sin 1°", *Araştırma*, vol. 3 (1965), p. 128.

[26] Smith, p. 626.

[27] Copernicus, p. 538.

[28] Smith, p. 617.

[29] Tekeli, *Araştırma*, vol. 3, p. 130

[30] *Ibid.*

1. If the sides of a triangle,

2. If two sides and an angle,

3. If a side and two angles.

are given the triangles are known.

The forth section of the first book of Taqî al-Dîn is on the plane Menelous theorem and plane triangles. In this section he mentions the important relation sin A/a = sin B/b = sinC/c and prooves it. While recognised by Al-Beyrûnî and Abû'l-Wafâ, Naṣîr al-Dîn who is the first, set it forth with clearness.[31]

As it is seen, Copernicus, in the calculation of plane triangles, repeats what Euclides said in the *Elements*.

*On the spherical triangles:*

The twelfth section of *De Revolutionibus* is on the spherical triangles.[32] At first he defines and gives the principal properties of a spherical triangle (that is to say the sides neither equal to nor greater than the halves of great circles)

In a spherical triangle having a right angle

1/2 ch 2AB/1/2 ch 2BC = radius 1/2 ch 2C,

that is to say,

sin AB/sinBC = radius/sin C

Beside this, spherical triangles having right angles,

1. If a side and an angle,

2. If three sides,

3. If three angles are given the triangles are known.

Equality of the spherical triangles follow this.

1. If in the same sphere, two triangles have right angles and another angle equal to another angle and one side equal to one side, they will have the remaining sides equal to the remaining sides and the remaining angle equal to the remaining angle.

2. If there is no right angle but provided that the sides which are adjacent to the equal angles are equal to one another. The triangles are equal to one another.

3. If two triangles in the same sphere have the sides of one severally equal to the sides of the other, they will have the angles of the one severally equal to the angles of the other.

[31] Smith, p. 630.

[32] Copernicus, p. 545-556.

4. If two triangles have two sides equal to two sides and an angle equal to an angle, whether the angle which the equal sides comprehend, or an angle at the base, they will also have the base and the remaining angles equal to the remaining angles.

The fifth and a part of the sixth section of the *Sidra al-Muntahâ* are on the spherical triangles and on the theorem of *Mugnî* atributed to Abû Naṣr ibn 'Irâq and on its conclusions. Taqî al-Dîn defines and gives the principal properties of a spherical triangles (that is to say the sides neither equal to nor greater than the halves of the great circles), and in a spherical triangle having a right angle, the theorem corresponding sides and angles. As Copernicus gives

$$\sin AB / \sin AC = \sin 90^\circ / \sin C$$

As it is known Abû Naṣr ibn 'Irâq (10th cen.) is the first who pointed this sine theorem and called it *Mugnî*.[33] Taqî al-Dîn gives the three following conclusions of *Mugnî*.

1. $\cos RY / \cos BR = \sin 90^\circ / \cos BY$ $\quad \triangle(RYB)$
2. $\cos B / \cos YR = \sin R / \sin 90^\circ$ $\quad \triangle(RYB)$
3. $\sin RY / \sin BY = \sin CR / \sin CH$ (the sides corresponding equal angles are proportional)

In the sixth section, in any triangle having obtuse or accute angles, he gives $\sin AB / \sin BC = \sin C / \sin A$ sine theorem.[34]

This important relation which is used frequently in finding the sides and the angles of any triangle is not mentioned by Copernicus.

In addition to this, he gives the following relations: In a spherical triangle if,

1. Two angles and a side adjacent to both angles,
2. Two sides and an angle comprehending one of these sides,
3. Two angles and a side which is adjasent to both angles,

are given, the triangle is known.

As it is seen, there is not only a quantitative but qualitative difference between Copernicus and Taqî al-Dîn who used the conclusions of the *Mugnî* and the sine theorem.

[33] Abû Naṣr Manṣûr ibn 'Alî ibn 'Irâq. Teacher of Beyrûnî. He gave in 1007-8 an improved edition of Menelaos's *Spherica*. Various other writings on trigonometry and astronomy are ascribed to him. Sarton, vol. 1, p. 668.

[34] Smith, p. 630.

*Taqî al-Dîn's work on tangent and cotangent:*

In the sixth section of the first book, Taqî al-Dîn defines tangent and the cotangent and gives the following fomulaes as

tg A/sin A = radius/cosA [35]

cotg A . tg A = radius$^2$ if r = 1 tg A . cotg A = 1

tg A/tg BC = sin A/sinAB that is to say tanget theorem, and he points that Abû'l-Wafâ al-Buzjânî is the first who applided this teorem.

Conclusions of the Tangent theorem:

1. cos B/sin 90° = cot BR/cot BY in triangle BYR
2. cos BR/sin 90° = cot B/tg R in triangle BRY
3. sin RY/tgYB = sin RH/tg HC in quadrilateral BHRC

While Taqî al-Dîn presents almost all the arguments for the tangents and cotangents, Copernicus says nothing on this subject.

[35] Abû'l-Wafâ knew this formulae. Smith. p. 623.

## ON THE STRAIT LINES IN A CIRCLE

*Copernicus*
*Book 1, Section 12*

The circle is divided into 360° and the diameter into 200,000$^p$ (p. 533).

*Theorem I.* The diameter of a circle being given, the sides of the triangle, tetragon, hexagon, and decagon, which the same circle circumscribes, are also given (p. 533).

*Porism.* Angle A and chord A are given, finding the chord (180°-A).

*Theorem II.* The theorem of Ptolemy (p. 534).

*Theorem III.* Finding the chord (A-B) (p. 535).

*Theorem IV.* Finding the chord A/2 (p. 535).

*Theorem V.* Finding the chord (A + B) (p. 535-536).

*Theorem VI.* arc. BC/arc AB >ch BC/chAB (p. 536).

*Problem.* But since the arc is always greater than the straight line subtending it-as the straight line is the shortest of those lines which have the same termini-nevertheless in going from greater to lesser sections of the circle, the inequality approaches equality, so that finally the circular line and the straight line go out of existence simultaneously at the point of tangency on the circle. Therefore it is necessary that just before

*Taqî al Dîn*
*Book 1, Section 1.*

The circle is divided into 360° and the diameter into 120$^p$ or 2$^p$ (7a).

*Theorem* I. II. III, IV, V. Finding the sides of a triangle, tetragon, hexagon and decagon inscribed in a circle, that is to say the chord 90°, chord 60°, chord 36° (7b, 8a). Angle A and the chord A are given, finding the chord (180°-A).

*Theorem* VI. The theorem of Ptolemy (8b-9a).

*Theorem* VII. Finding the chord (A-B) (9a).

*Theorem* VIII. Finding the chord A/2 (9a).

*Theorem* IX. Finding the chord (A + B) (9b).

*Theorem 1.* arc BC/arc AB > ch. BC/ch. AB (7b).

*Theorem 10.* It is easy to obtain the lenghts of chords of certain arcs. For others, formulas are needed as ch 2A, ch A/2, ch (A-B), ch (A + B). Even by applying all these formulas it is not possible to get the chord 1° or chord 2°. "The Ancients could not find a correct way, in consequence of this, they depended on an approximate method which is not worth to describe. The Late Ulug Beg said, 'We had inspriation about extracting chord 1°

that moment they differ from one another by no discernible difference.

Let arc AB = $3°$

and arc AC = $1\ 1°/2$

ch AB = 5235 (diameter = 200,000)
ch AC = 2618.

And though arc AB < 2 arc AC

Yet ch AB = 2 ch.AC

and ch AC = 2617 = 1

But if we make arc AB = $1\ 1°/2$

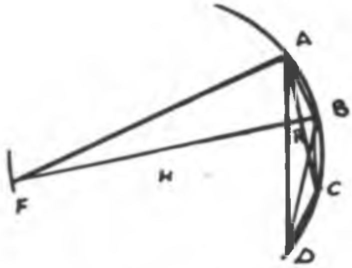
and arc AC = $3/4°$

then ch AB = 2618

and ch AC = 1309

and even though chord AC ought to be greater than half of chord AD, it is seen to be no different from the half. And the ratios of the arcs and straight lines are now apparently the same. Therefore, since we see that we have come so far that the difference between the straight and the circular line evades senseperception as completely as if there were only one line, we do not hesitate to take 1309 as subtending $3/4°$ and in the same ratio to fit the chord to the degree and to the remaining parts (of the degree), and so with the addition of $1/4°$ to the $3/4°$ we establish $1°$ as subtended by 1745, $1/2°$ by 872 1/2, and sin $1°$.' He wrote a text on this subject and explained three ways of finding the chord $1°$ and sin $1°$, depending the geometric theorems related with mathematics. (10b).

One of them is as follows."




(Figure 1)

$$AD = ch\ 6° = 6^p16'49''7'''59''''8'''''56''''''20'''''''$$

$$AB = ch\ 2° = X$$

According to Ptolemy's theorem

$$AC^2 = X^2 + X.AD \quad \boxed{I}$$

As the triangle BAF is a right triangle, and AR is a perpendicular drawn to the hypotenuse,

$$X^2 = BR.BF \qquad BF = diameter = 2$$

$$BR = X^2/2 \quad \boxed{II}$$

ABR is a right triangle so,

$$X^2 = BR^2 + AR^2$$

$$X^2 = X^4/4 + AR^2$$

On the other hand

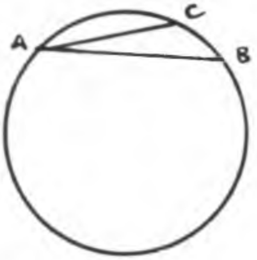
$$AR^2 = 1/4\ AC^2$$

$$X^2 - X^4/4 = 1/4\ AC^2$$

$$4X^2 - X^4 = AC^2$$

$$4X^2 - X^4 = X^2 + X.AD$$

and 1/3° by approximately 582 (p. 537-538).




(Figur 2)

According to I and III

$$3X = X^3 + AD$$

$$X = \frac{X^3 + AD}{3}$$

As it does not belong to one of the six equations, so he tried to solve as follows.

$X = AD/3 = 2^p 5'36''22'''39''''42'''''58''''''46'''''''$

In reality

$$X = \frac{(a + 2p\,5'36''\ldots)^3 + AD}{3}$$

As he keeps on doing this till very small changement occurs, he gets

ch. $2° = 2^p 5'39''26'''22'''' 29'''''32''''''$ (10a-10b).

## ON THE SINES

"Nevertheless I think it will be enough if in the table we give only the halves of the chords subtending twice the arc, whereby we may concisely comprehend in the quadrant what it is used to be necessary to spread out over semicircle, and especially because the halves come more frequently into use in demonstration and calculation than the whole chords do. Now we have set forth a table increasing by 1/6°'s and having three columns. (p. 538).

*Book I, Section 2. on the sines:*

*Sine* of an arc is a perpendicular drawn from one end of the arc to the line joining the other end of the arc to the centre. It is the half of the cord subtending twice the angle.

*Cosine* is the segment between the perpendicular and the centre.

*Sehm* is the segment of the radius between the perpendicular and the arc.

As the diameter is greater than the chord and in limit the chord 180° is equal to the diameter, the radius is greater than the sine and the sin 90° is equal to the radius.

*Theorem XIII. Finding sin 18°, sin 36°.*

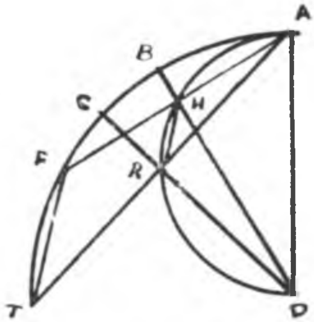
AB is bisected at C, AC at D, DC at H. RC, equal to DC, is drawn from C perpendicular to DC. RH be joined. HR=HF

Therefore DF is cut at C in extreme and mean ratio.

so DC = sin15°, CF = sin18° AB being the diameter.

*Theorem XV. Finding sin (A–B):*

Given : arcAB, arcAC and sinAB = AH, sinAC = AR




(Figure 3)

Wanted : sinBC = RH

The circle having the diameter AD passes on the points H and R.

Since ARHD is a quadrileteral inscribed in a circle,

so AR.DH = AH.RD + RH.DA AH = sinAB,

AR = sinAC, HD = cosAB, RD = cosAC

AD = $60^p$, HR = sin (AC–AB)

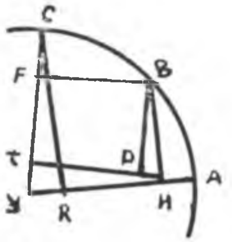
Then $$\text{Sin } (AC-AB) = \frac{\text{sin}AC.\text{cos}AB-\text{sin}AB.\text{cos}AC}{60^p}$$

*Theorem XVI. Another way.*

Given : arcAB, arcAC, and sinAB, sinAC

Wanted : sinBC that is to say sin (AC–AB)

YC ⊥ BF, BF // HT and R = T = 90°, angleY is common




(Figure 4)

therefore $\triangle YCR \sim \triangle HYT$

and $YH/HT = YC/CR$ YH is given, YC = the diameter, YH = cosAB

so HT is known

As BD // CY and BH // CR so angle HBD = angle YCR and angle D = angleY

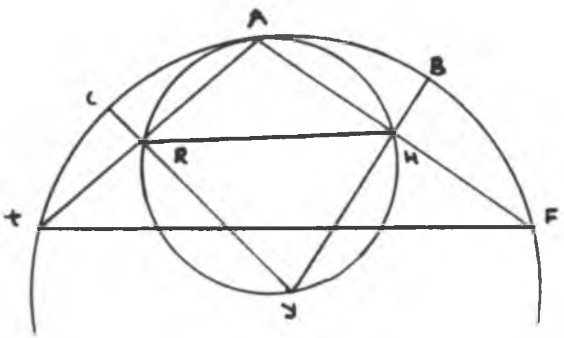
Therefore : $\triangle BHD \sim \triangle YCR$ and $BH/HD = CY/YR$

As BH,CY,RY are given, HD is known

HT–HD = DT On the other hand $DT \perp CY$, $BF \perp CY$

DT = BF and BF = sin (AC–AB)

$$\sin (AC\text{-}AB) = \frac{\cos AB . \sin AC - \sin AB . \cos AC}{60^p} \quad (11b)$$




(Figure 5)

*Theorem XVII. Finding sin (A + B).*

Given : ArcAB, arcAC and sinAB = AH, sinAC = AR
Wanted : sin (AB + AC) = HR

The circle having the diameter AY passes on the points H and R

HR = 1/2FT and HR = sin (AB + AC)

because FT = 2 ch. (AB + AC)

In the quadrilateral AHYR, AR.HY + AH.RY = RH.AY

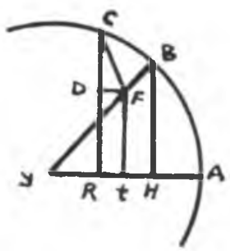
AR = sinAC, HY = cosAB, AH = sinAB, RY = cosAC,

AY = $60^p$, RH = sin (AB+AC)

So $$\sin(AB + AC) = \frac{\sin AC.\cos AB + \sin AB.\cos AC}{60^p} \quad (12a)$$

*Theorem XVIII. Another method.*

arcAB, arcBC, and sinAB=BH, sinBC = CF are given




(Figure 6)

sin (AB + BC) = CR is wanted.

$\triangle$BYH ~ $\triangle$FYT because AY ⊥ BH, AY ⊥ CR, AY ⊥ FT, CR ⊥ FD

and BH/FT = BY/YF

On the other hand FT = DR and DR is given.

As $\triangle$FCD ~ $\triangle$BYH ~ $\triangle$FTY

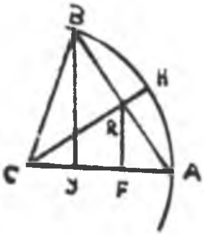
FC/CD = BY/YH FC,BY,YH are given so CD is known.

CD + DR = CR that is to say CR = sin (AB + BC)

$$\sin(AB+BC) = \frac{\sin AB.\cos BC + \sin BC.\cos AB}{60^p} \quad (12a)$$

*Theorem XIX. Finding sin A/2.*

$\sin AB = BY$ is given.




(Figure 7)

$\sin 1/2 AB = BR$ is wanted.

$RF \perp AC$ and AY is divided by RF in equal parts.

As YC is given $AY = 1 - \cos AB$, AF and AR are known.

$AR = \sqrt{AF^2 + RF^2}$ $\quad AF = \text{ver.}\sin A$ and $RF = \sin A$

So $\quad \sin AB/2 = \sqrt{\dfrac{\text{ver } \sin^2 AB + \sin^2 AB}{4}}$ (12b)

*Theorem XX. Finding sin 1°.*

$X = \sin 1°$, $\quad BF = {}^{1}$ or $BF = 60^{p}$

ABF is a right triangle and $AR \perp FB$

so $\quad AB^2 = BR.BF$ if $BF = 1$ and

$AB = \sin 1° = X$

$X^2 = BR$ and $X^4 = BR^2 \ldots$ [I]

In the quadrilateral ABCD

$X.AD + X^2 = AC^2$

$AC = 2AR$ and $AC^2 = 4AR^2$,

$X.AD + X^2 = 4AR^2$

$\dfrac{X.AD + X^2}{4} = AR^2$ ........... [II]

ABR is a right triangle $\quad AB^2 = AR^2 + BR^2$, $\quad X^2 = AR^2 + BR^2$

$X^2 = X^4 + AR^2$ according to [I]

$X^2 = X^4 = 1/4\ AD + 1/4\ X^2$

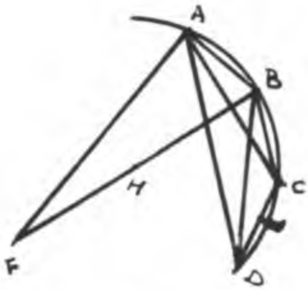
$X = \dfrac{AD + 4X^3}{3}$

Approximately $X = AD/3 = 1/3\ 3^{p}8'24''36'''59''''35''''' 28''''''15'''''''$

Hovewer $X = a + 1/3\ 3^{p}8'24''\ldots\ldots$

The formulae $X = \dfrac{4(a+2^{p}\ldots\ldots)^{3} + AD}{3}$

He keeps on doing this till very litle changement occurs. At last he gets $\sin 1° = 1^{p}2'48''11'''19''''51'''''29''''''25'''''''$ (12b-13a)




(Figure 8)

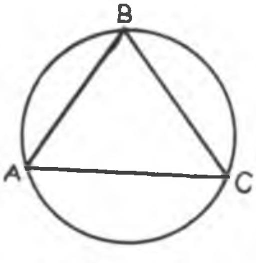
## ON THE PLANE TRIANGLES

*Book 1. section 15.*

I

The sides of a triangle whose angles are given are given.

Let there be the triangle ABC,




(Figure 9)

*Bock I, section 4.*

*Theorem XXX:* XXX Let there be the triangle around which a circle circumscribed (it may be acute, right, obtuse).

The sides of the triangle are proportional to the sines of the angles, subtending the sides (Figure 10).

AB/BC = sin C/sin A, Theorem of sines. Let YB be joined and let perpendiculars YH and YR be dropped on AB and BC. In thefigure d, the perpendicular is extended to the point F.

around which a circle is circumscribed.

Therefore arcs AB, BC, and CA will be given in degrees.

II

If two sides of a triangle are given together with one of the angles, the remaining side and the remaining angles may become known (p. 543).

III

If the angle BAC comprehended by the given sides is right, the same thing will result.

IV

If the given angle ABC is acute, the same thing will result.

V

If the angle ABC is obtuse, the same thing will result.

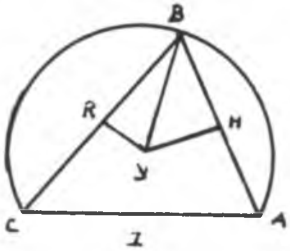
VI

Given all the side of the triangle, the angles are given.

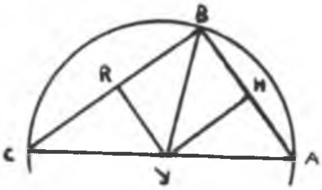
BYH = C
BYR = A
BYF = A
BH = sin 1/2 arc AB = sin C
BR = sin 1/2 arc BC = sin A

As AB/BC = BH (sinBYH=sin C)/BR (sin BYR=sinA)

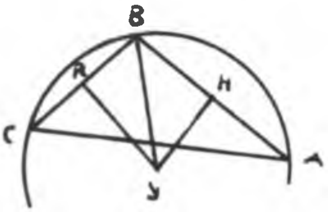
so AB/BC = sin C/sin A (16a)



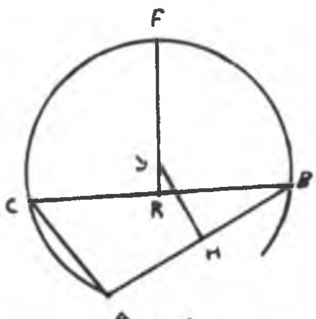

a




b




c




d

Figure 10

## SPHERICAL TRIANGLES

*14. On the spherical triangles:*

I

If there are three arcs of the great circles of a sphere, and if any two of them joined together are longer than the third a spherical triangle can be constructed from them. (p. 545)

II

The arcs of the spherical triangle must be less than a semicircle. (p. 546).

III

In spherical triangles having a right angle, the chord subtending twice the side opposite the right angle is to a chord subtending twice one of the sides comprehending the right angle as the diameter of the sphere is to the chord which subtends the angle comprehended in the great circle of the sphere by the first side and by the remaining side.

Let there be spherical triangle ABC and $C = 90°$

ch2AB/ch2BC = dmt. sph./ ch 2 BAC

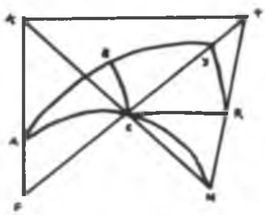
With A as a pole drawn DE the arc of a great circle, and let ABD and ACE the quadrants of the circles be completed. And from the centre F of the sphere draw the common sections of the

*Section 5, on the Mugnî attributed to Ebû Naṣr ibn 'Irâq and on its conclusions.*

*Theorem* XLI.

Let there be the spherical triangle ABC. The sides neither equal to nor greater than the half of the great circle.

$B = 90°$



(Figure 11)

sinA/sinBC = sinB/sin AC

Because: Let the sides AB and AC extended up to the points Y and H. With C as a pole let arc YH of the great circle be described. Let $A = YH$, $CB = YR$

and plane CR // the plane of the circle AY.

Let the chord HR and the line FY, on the other hand the line HC and FA be extended, until they cut one another at the points T and K respectively.

The line KT is on the surface of the circle AY and on the triangle HRC.

And KT // RC

circles: FA the common section of circles ABD and ACE, FE of circles ACE and DE, and FD of circles ABD and DE, and FC of the circles AC and BC. Then draw BG at right angles to FA, BI at right angles to FC, and DK at right angles to FE, and let GI be joined.

angle AED = angle ACB = 90°
plane EDF ⊥ plane BCF ⊥ plane

AEF
KD ⊥ cirle AEF
KD // BI, FD // GB
angle FGB = angle GFD = 90°
angle FDK = angle GBI
angle FKD = 90°
So GI ⊥ IB

The sides of similar triangles are proportional, and

DF/BG = DK/BI

But BI = 1/2 ch. 2 CB
BG = 1/2 ch. 2 BA
DK = 1/2 ch. 2DE = 1/2 ch. 2 DAE
DF = 1/2 dmt. sph.

Therefore

ch. 2AB/ch. 2 BC = dmt./ch. 2 DAE. IV, V on finding the sides or the angles of the spherical triangles having right angles.

XIII

All the sides of a triangle being given, the angles are given. The

In the triangle HKT

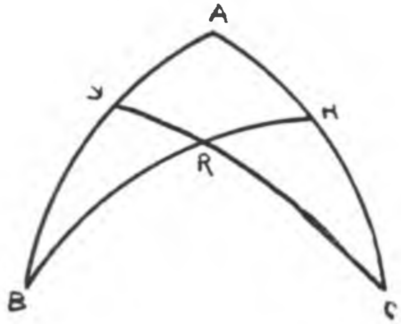
HT/TR = HK/KC

HT/TR = sinHY/sinYR = sinA /sinBC. HK / KC=sinHA (90°) sin CA = sin 90° / sin CA
So sinA / sinBC = sin 90° / sinCA

*Theorem* XLIII. *The first conclusion of the Mugnî:*

AC = CY = BH = BA = 90°
CA and CY, BH and BA cut one another at the points R and A respectively.

B is the pole of arc AC and C is the pole of arc AB




(Figure 12)

cosYR/cosBR = sin 90°/cosBY

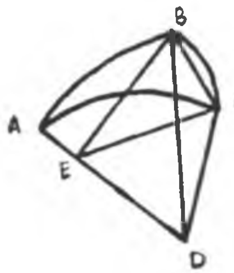
*The first conclusion of the Mugnî.*
Because: If in place of angle B, the angle C is assumed,
sin CH/sin CR = sin R/sin 90°
As sinCR = cos B
sin HR = cos BR
sin C = sin AY = cos BY
If the equals be substituted, the first result of the *Mugnî* is obtained (19a).

sides of triangle ABC are given. I. side AB = side AC (p. 554).

1/2 ch. 2 AB = BE

and 1/2 ch. 2 AC = CF which on account of being at equal distance from the centre of the sphere will cut one another at point E in DE the common section of the circles. In plane ABD angle DEB = 90°
and in plane ACD angle DEC = 90°

(Euclid XI, 3) angle BEC is the angle of inclination of the planes. As the sides of rectilinear triangle BEC are given on account of their arcs being given. Angle BEC given, so the sides of the triangle ABC.



(Figure 14)

II. If AC > AB
1/2 ch. 2 AC = CF will

fall lower down.

But if AC < AB CF will fall higher up.

Let FG // BE and at point G

*Theorem* XLIV *The second result of the Mugnî:*

Let there be the spherical triangle YBR.

cosB/cosYR = sinR/sin90°

If in place of angle B, the angle C is assumed.

sinCH/sinCR = sinR/sin90°

sinHC = cosAH (the arc subtending the angle B)

sinCR = cosYR

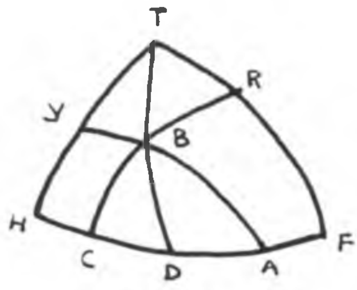
If the equals be substituted the second result of the *Mugnî* is obtained.

*Theorem XLV. The third result of the Mugnî:* The spherical triangle BYR ∼ the spherical triangle CHR

Because H = Y = 90°
and R = R
so sinRY/sinBY = sinHR/sinCH
This is the third result of the *Mugnî:*

*Section 6th on the ratios of the sides of the triangles.*

*Theorem LII.* Let there be the spherical triangle ABC (It is either right or acute or obtuse)



(Figure 13)

let it cut BD the common section of the two circles (AB and BC)

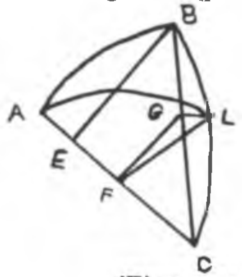
EFG = AEB = EFC=90°

CF = 1/2 ch. 2 AC

Therefore angle CFG will be the angle of section of circles AB and AC, and the angle is known too.

As △DFG ∼ △DEB FG is given in the parts wherein FC is also given and DG/DB = DE/EB

DG will be given in the same parts whereof DC has 100,000. But as the angle GDC is given through the arc BC, therefore by the second theorem on plane triangles the sides GC is given in the same parts wherein the remaining sides of the plane triangle GFC are given. Therefore by the last theorem on plaine triangles angle GFC, the spherical angle BAC, is given (p. 554-555)




(Figure 15)

XV

If all the angles of a triangle are given, even though none is right angle, all the sides are given. The angles of triangle ABC are given.

sinAB/sinBC /= sinC/sinA

Let sides AB and AC be completed into quadrants. T is the pole of the arc CA. Let the great circle TBD cut the sideAC at the pointD.

In the quadrilateral TFCB,
sinTD/sinDB = sinTF/sinFR.
sin RC/sinCB
and as sinTD = sinTF
Then sinRF.sinCB=sinRC.sinBD
In the quadrilateral THAB,
sinTD/sinDB = sinTH/sinHY.
sinYA/sinAB

As sinTD = sinTH

so sinHY.sinAB = sinAY.sinBD

As sinRC = sinAY.

so sinRF.sinBC=sinYH.sinBA

Depending on *Mugnî*,

sinRF/sinBD=sinRC/sinBC

and sinBD/sinYH=sinAB/sinAY

As AY = RC

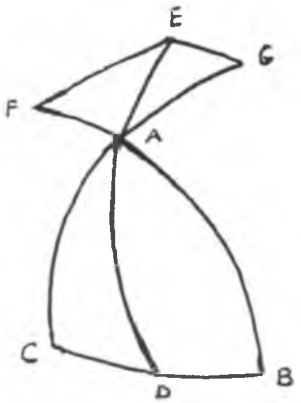
so sinAB.sinYH (sinA) = sin BC.sinRF (sinC)

That is to say sinAB/sinBC = sinC/sinA

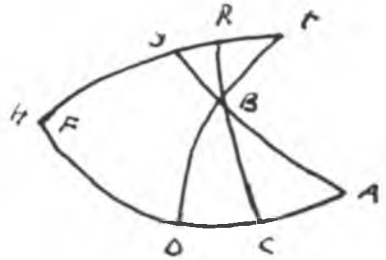
*Theorem LIII.*

Let the angle C be an obtuse angle, so the great circle TBD cuts the side AC at its extention. In this situation, the points F and H coincide. The same things follow as before.

$AD \perp CB$, $CAT = DAE = BAF = 90^\circ$ $F = G = 90^\circ$
Therefore in the right triangle EAF 1/2 ch. 2 AE/1/2 ch. 2 EF = 1/2 dmt. sph./1/2 ch. 2EAF.
Similary in right triangle AEG 1/2 ch. 2 AE/1/2 ch. 2EG = 1/2 dmt. sph./1/2 ch. 2EAG
So 1/2 ch. 2EF/1/2 ch. 2EG = 1/2 ch 2EAF/1/2 ch. 2EAG
And because arcs FE and EG are given,
since arcFE = $90^\circ$ – angle B
and arcEG = $90^\circ$ – angle C
Thence the ratio angles EAF and EAG given, i.e., the ratio between BAD and CAD, which on their vertical angles. The whole angle BAC has given; therefore by the foregoing theorem, angles ABD and CAD will also be given. Then by the fifth theorem we shall determine sides AB, BD, AC, CD, and the whole arc BC.



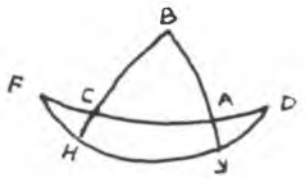

(Figure 18)




(Figure 15)

*Theorem LIV.*
Let the triangle ABC be given-the sides are smaller than quadrants and the angles are smaller than $90^\circ$. With B as a pole let a great circle be drawn. The extention of the side AB cuts this arc at the point Y, and the extention of the arc BC at the point H, and the extention of the arc AC at the points F and D.

arc AY and CH are known
and DA + CF
so sinDA/sinAY = sinFC/sinHC
As sinDA + sinCF
and sinDA.sinAY=sinCF.sinCH.
If the plases are changed,
sinDA/sinCF=sinAY/sinCH
Since sinAY sinCH is given.




(Figure 16)

and $\frac{\sin DA + \sin CF}{\sin CF}$ as sin DA+sin CF is given

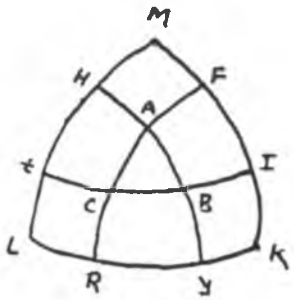
so arcAD and arcCF are known. By applying the *Mugnî* theorem the angles of the triangles are obtained.

*Theorem LV.*

Let there be the triangle ABC having the preceding qualities. On account of the angles being given all the sides of the triangle are given. Let the sides be extended and completed into quadrants. With A, B, and C as poles let the circles be drawn, intersecting each other at the points K, L, M

arcRY = angleA
arcTH = angleB
arcFI = angleC
As arcIK + arcFI=90°
and arcFM + arcFI=90°
so the arcKM is known.

Similarly arcKL and arcLM are known.




(Figure 17)

## THEOREM OF SHADOW

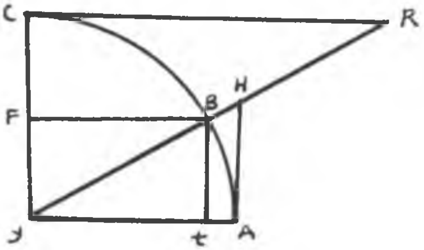
Copernicus says nothing on this subject.

*Section six on shadow theorem and the conclusions derived from it.*

*Umbra Versa, tangent,* is a straight line that is perpendicular to the radius (one side of the angle) and touches the circle and cuts the extention of the other side of the angle, parallel to its sine.

*Umbra recta, cotangent,* is a straight line that is perpendicular to the radius, and touches the circle and cuts the extention of the other side of the ange, parallel to its cosine.

*Theorem XLVII:* Let there be the quadrant ABC around the centre Y to explain the properties of shadow. Let AY, YC, and YB be joined and YB be extended to the point R. Let perpendiculars be erected on the points A and C, cutting the extension BY at the points H and R.




(Figure 20)

CR = the first shadow, umbra versa, tangent

BF (sinBC) // CR(tg BC)

AH = tg AB

BT(sin AB) // AH(tg AB)

$\triangle AHY \sim \triangle TBY \sim \triangle CRY \sim \triangle FYB$

tg A/sin A = radius/sin (90°–A)

Because HA/AY (CY) = YC/CR

HA = tgA, AY = CY = yarıçap, CR = tg (90°–A) = ctg A

So tg A/ctg A = $radius^2$ if radius = 1

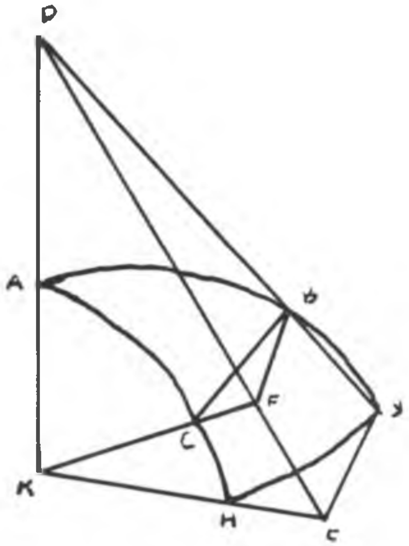
tg A. ctg A = 1

*Theorem XLVIII: Theorem of Tangent atributed to Abû'l-Wafâ al-Buzjânî:*

Let there be a spherical triangle ABC comprehended by the arcs of great circles,

and B = 90°

tg A/tg BC = sin B (90°)/sin AB




(Figure 21)

Let the sides AB and AC be extended to the points Y and H. Let the centre R of the sphere, A and C and H be joined. Let the perpendiculars BF and Yc dropped to the plaine of the circle ABY and they cut RE and RF at the point F and c.

$$BF = tgBC, \qquad Yc = tg\ YH$$

Let Y, B be joined and be extended and cut DF at the point D. c,F,D are on a line,

$$\overset{\triangle}{YcD} \sim \overset{\triangle}{BFD}$$

$$tgA(Yc)/tg\ BC(BF) = YD/DB = \sin AY\ (\sin 90^\circ)/\sin AB.$$

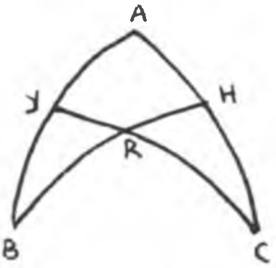
So $\quad tgA/tgBC = \sin 90^\circ/\sin AB \qquad$ (according I, 32)

*Theorem XLIX. The first conclusion of tangent theorem:* Let there be the quadrilateral BACR. Let B be the pole of arc AC and C be the pole of arc AB.

$$\cos B/\sin 90^\circ = ctgBR/ctgBY$$

In triangle HCR

$$\sin CH\ (\cos B)/\sin AC\ (\sin 90^\circ) = tg\ RH\ (ctg\ BR)/tgc\ (ctg\ BY)$$




(Figure 22)

*Theorem L. The second conclusion of tangent theorem:* In the same triangle $\cos BR/\sin 90° = \mathrm{ctg} B/\mathrm{tg} R$ (Figure 22)

Because in same quadrilateral

$\cos BR/\sin 90° = \mathrm{tg}\ CH\ (\mathrm{ctg}\ B)/\mathrm{tg}\ R = \mathrm{ctg}\ B/\mathrm{tg}\ R$

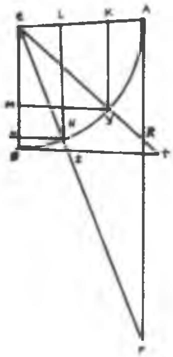
*Theorem LI. The third conclusion of tangent theorem:* In the same quadrilateral $\sin RY/\mathrm{tg}\ YB = \sin RH/\mathrm{tg}\ HC$ (Figure 22)

Because $\triangle YRB \sim \triangle RHC$

so $\sin RY/\mathrm{tg}\ YB = \sin RH/\mathrm{tg}\ CH$

*One of the pecularities of the shadow:* If $AY < AH$ (Figure 23)

$\mathrm{tg}\ AY\ (AR)/\mathrm{tg}\ AH (AF) = \mathrm{ctg}\ AH\ (BI)/\mathrm{ctg}\ AY (BT)$



(Figure 23)